SIMPLIFYING PURE MATHEMATICS 1

BY

MUKOOBA JOEL, B. SCE (HONS)

BUSITEMA UNIVERSITY

Sales department: Mr. Gantungo Vicent greenfields high school Iganga

Contacts: 0773791369, 0755501430

PREFACE

This book is a thorough description of pure mathematics as per the current syllabus of Uganda Advanced Certificate of Education.

The content is divided per class and term as described in the syllabus.

This is a senior five volume called simplifying pure mathematics 1. The one for senior six is called simplifying pure mathematics 2.

All chapters are incorporated in this volume and much emphasis is made to the examination techniques of how to answer the questions.

First Edition 2021

CONTENTS

SENIOR FIVE

TERM ONE

SENIOR FIVE: TERM ONE

CHAPTER 1

INDICES, LOGARITHMS AND SURDS

1.1. Indices

- An index is a power to any given number. i.e. x^y, y is a power/index/exponent and x is the base.
- The following are the laws of indices;

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $(a^m)^n = a^{mn}$

Where a, m and n are real numbers and $a \neq 0$

- From the above laws we can deduce that

(i) $a^0 = 1$ (ii) $a^{-n} = \frac{1}{a^n}$ (iii) $a^{\frac{1}{n}} = \sqrt[n]{a}$ (iv) $a^{m/n} = \left(\sqrt[n]{a}\right)^m$

Example 1

1. Simplify the following;

(a) $2^3 \times 2^4$
(b) $5^7 \div 5^2$
(c) $3^{-2} \times 3^5 \div 3^{-2}$
(d) $\left(\frac{32}{243}\right)^{3/5}$
(e) $49^{1.5}$

Solution

(a) $2^3 \times 2^4 = 2^{3+4}$
$= 2^7$
(b) $5^7 \div 5^2 = 5^{7-2}$
$= 5^5$
(c) $3^{-2} \times 3^5 \div 3^{-2} = 3^{-2+5-(-2)}$
$= 3^5$
(d) $\left(\frac{32}{243}\right)^{3/5} = \left(\frac{2^5}{3^5}\right)^{3/5}$
$= \left(\frac{2}{3}\right)^3$
(e) $49^{1.5} = (7^2)^{3/2}$
$= 7^3$

Example 2

Solve the following equations;

(a) $3^x = \frac{1}{27}$
(b) $2^x = 32$
(c) $3^{x+2} = 81$
(d) $2^{2x} - 3(2^x) + 2 = 0$
(e) $2^{2x+1} + 5(2^x) - 3 = 0$

Solution

(a) $3^x = 3^{-3}$; $\underline{\therefore x = 3}$
(b) $2^x = 2^5$; $\underline{\therefore x = 5}$
(c) $3^{x+2} = 3^4$
$\Rightarrow x + 2 = 4$; $\underline{\therefore x = 2}$
(d) $(2^x)^2 - 3(2^x) + 2 = 0$
Let $2^x = y$
$\Rightarrow y^2 - 3y + 2 = 0$
$\Rightarrow y^2 - y - 2y + 2 = 0$
$\Rightarrow y(y-1) - 2(y-1) = 0$
$\Rightarrow (y-1)(y-2) = 0$
Either $y = 1$; $2^x = 1$; $2^x = 2^0 \therefore x = 0$
Or $y = 2$; $2^x = 2^1$; $\therefore x = 1$
$\underline{\therefore\text{the solutions are } x = 0 \text{ and } x = 1}$
(e) $2^{2x}.2^1 + 5(2^x) - 3 = 0$
$\Rightarrow 2(2^x)^2 + 5(2^x) - 3 = 0$
Let $y = 2^x$
$\Rightarrow 2y^2 + 5y - 3 = 0$
$\Rightarrow 2y^2 + 6y - y - 3 = 0$
$\Rightarrow 2y(y+3) - 1(y+3) = 0$
$\Rightarrow (2y-1)(y+3) = 0$
Either $2y = 1$; $2^x = 1/2$; $2^x = 2^{-1}$; $\therefore x = -1$
Or $y = -3$; $2^x = -3$ NA
$\underline{\therefore\text{the only solution is } x = -1}$

Example 3

Simplify the following;

(a) $\frac{(1+x)^{1/2} - \frac{1}{2}x(1+x)^{-1/2}}{(1+x)}$

(b) $\frac{\sqrt{(1-x)}\frac{1}{2}(1+x)^{-1/2} + \frac{1}{2}(1-x)^{-1/2}\sqrt{(1+x)}}{(1-x)}$

Solution

(a) $\frac{(1+x)^{1/2}-\frac{1}{2}x(1+x)^{-1/2}}{(1+x)} = \frac{\frac{(1+x)^{1/2}}{1}-\frac{x}{2(1+x)^{1/2}}}{(1+x)}$

$= \frac{2(x+1)-x}{2(1+x)^{3/2}}$

$= \frac{2x+2-x}{2(1+x)^{3/2}}$

$= \frac{x+2}{2(1+x)^{3/2}}$

(b) $\frac{\sqrt{(1-x)}\frac{1}{2}(1+x)^{-1/2}+\frac{1}{2}(1-x)^{-1/2}\sqrt{(1+x)}}{(1-x)} = \frac{\frac{\sqrt{(1-x)}}{2(1+x)^{1/2}}+\frac{\sqrt{(1+x)}}{2(1-x)^{1/2}}}{(1-x)}$

$= \frac{1-x+1+x}{2(1-x)^{3/2}(1+x)}$

$= \frac{2}{2(1-x)^{3/2}(1+x)}$

$= \frac{1}{2(1-x)^{3/2}(1+x)}$

Example 4

Find the value of x from the equation $\frac{6^5 \times 6^x}{36} = 6^9$

Solution

$\Rightarrow \frac{6^{5+x}}{6^2} = 6^9$

$\Rightarrow 6^{3+x} = 6^9$

$\Rightarrow 3 + x = 9;$

$\underline{\therefore x = 6}$

Example 5

If $a^x = b^y = c^z$ and $b^2 = ac$ show that $y = \frac{2xz}{x+z}$

Solution

$\Rightarrow a = b^{y/x}$and$c = b^{y/z}$

$\Rightarrow b^2 = b^{y/x}.b^{y/z}$

$\Rightarrow b^2 = b^{\left(\frac{y}{x}+\frac{y}{z}\right)}$

$\Rightarrow \frac{y}{x} + \frac{y}{z} = 2$

$\Rightarrow \frac{y(x+z)}{xz} = 2$

$\therefore y = \frac{2xz}{x+z}$…………………■

Example 6

Solve each of the following equations;

(a) $x^{\frac{1}{5}} = 3$ (b) $x^{4/3} = 81$ (c) $2x^{3/4} = x^{1/2}$ (d) $x^{1/3} - 3 = 28x^{-1/3}$

Solution

(a) $\left(x^{\frac{1}{5}}\right)^5 = 3^5 \therefore x = 243$

(b) $\left(x^{4/3}\right)^{3/4} = (81)^{3/4}$

$\Rightarrow x = 27$

(c) $\left(2x^{3/4}\right)^4 = \left(x^{1/2}\right)^4$

$\Rightarrow 2^4(x^3) = x^2$

$\Rightarrow x^2(16x - 1) = 0$

$\underline{\therefore x = 0, x = 1/16}$

(d) $\Rightarrow x^{\frac{1}{3}} - 3 = \frac{28}{x^{\frac{1}{3}}}$

Let $y = x^{\frac{1}{3}}$

$\Rightarrow y - 3 = \frac{28}{y}$

$\Rightarrow y^2 - 3y - 28 = 0$

$\Rightarrow y^2 - 7y + 4y - 28 = 0$

$\Rightarrow y(y - 7) + 4(y - 7) = 0$

$\Rightarrow (y + 4)(y - 7) = 0$

$\Rightarrow y = -4;\ x^{\frac{1}{3}} = -4;\ \therefore x = -64$

$\Rightarrow y = 7;\ x^{\frac{1}{3}} = 7; x = 343$

$\therefore$ the solutions are $x = -64$ and $x = 343$

Exercise 1.1

1. Solve the following equations

(a) $9^x = 27$ (b) $5^{x+3} = 1$ (c) $2^{x-3} = 4^{x+1}$ (d) $3^x.3^{x-1} = 9$

2. Solve the following equations;

(a) $2^{2x} - 5.2^x + 4 = 0 Ans(x = 0, 2)$

(b) $9^{x+1} - 3^{x+3} - 3^x + 3 = 0 Ans(x = -2, 1)$

(c) $2^{2x} - 9.2^x + 8 = 0 Ans(x = 0, 3)$

(d) $3^{2x} - 10.3^x + 9 = 0 Ans(x = 0, 2)$

(e) $16^x - 5.2^{2x+1} + 1 = 0 Ans(x = 1/2, -1/2)$

(f) $4^x - 3.2^{x+1} + 8 = 0 Ans(x = 1, 2)$

3. Simplify the following expressions;

(a) $x^2\sqrt{\left(1 - \frac{1}{x^3}\right)}$ (b) $\frac{1}{x}\sqrt{(x^2 + x^4)}$ (c) $(x - 2)^{3/2} + 2(x - 2)^{1/2}$

(d) $(2x - 1)^{-1/2} + (2x - 1)^{1/2}$

4. Solve the following equations simultaneously
   $2^x + 3^y = 5; \quad 2^{x+3} - 3^{y+2} = 23. \qquad Ans(x = 2, y = 0)$
5. Solve $\frac{16^x - 4^x}{4^x + 2^x} = 5(2^x) - 8$
6. Express with positive indices;

(a) $\frac{x^{-2}y^3z^{-4}}{6} \times \frac{9}{x^3y^{-3}z^4}$ (b) $\frac{\sqrt[3]{abc^{-4}}}{\sqrt[4]{a^3b^{-3}c}}$

7. Show that $\sqrt{x} - \sqrt{a} = \frac{x-a}{\sqrt{x}+\sqrt{a}}$. Deduce that $\frac{1}{\sqrt{x}-\sqrt{a}} = \frac{\sqrt{x}+\sqrt{a}}{x-a}$
8. Simplify $(4.2^{n+1} - 2^{n+2})/(2^{n+1} - 2^n)$
9. Solve the following equations

(a) $27^{x-3} = 3 \times 9^{x-2}$ (b) $2^{2+2x} + 3 \times 2^x - 1 = 0$ (c) $9^{x^2} = 3^{5x-2}$

(e) $2^{x^2} = \frac{1}{4}8^x$ (e) $4^{2x} = 2^{6x-1}$ (f) $8 \times 2^x = \left(\frac{1}{4}\right)^{3x+5}$

10. Solve equations simultaneously: $3^{2x+y} = 12$, $2^{x-y} = 4$
11. Solve each of the following equations for x;

(a) $3x^3 = 375$ (b) $98x^2 = 2$ (c) $x^3 + 343 = 0$ (d) $\frac{2}{49}x^{-2} + 14x = 0$

(b) $\frac{9}{25}x = \frac{5}{3}x^{-2}$

12. Solve each of the following equations;

(a) $x^{2/3} - x^{1/3} - 2 = 0$ (b) $2x^{1/4} = 9 - 4x^{-1/4}$ (c) $x^3 + 8 = 9x^{3/2}$.

13. Solve the equation $\frac{16^x - 4^x}{4^x + 2^x} = 5(2^x) - 8$

1.2. Surds

- A surd is a number of the form $b\sqrt{a}$ where a is not a perfect square
- Surds can be multiplied and divided in the form;

(i) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ (ii) $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$ But $\sqrt{a} + \sqrt{b} \neq \sqrt{(a+b)}$

- Two surds are said to be conjugate of each other if their product is a rational number. For example $\sqrt{2}$ and $\sqrt{2}$ are conjugate surds because $\sqrt{2} \times \sqrt{2} = \sqrt{4} = 2$.
  Similarly $2 - \sqrt{3}$ and $2 + \sqrt{3}$ are conjugate of each other because
  $(2 - \sqrt{3})(2 + \sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4 - 3 = 1$
- Rationalizing the denominator of a fraction means making it a rational number.

Example 7

Simplify $3\sqrt{2} + 5\sqrt{3} + 7\sqrt{2} - \sqrt{3}$

Solution

$$\begin{aligned} 3\sqrt{2} + 5\sqrt{3} + 7\sqrt{2} - \sqrt{3} &= 3\sqrt{2} + 7\sqrt{2} + 5\sqrt{3} - \sqrt{3} \\ &= 10\sqrt{2} + 4\sqrt{3} \\ &= 2(5\sqrt{2} + 2\sqrt{3}) \end{aligned}$$

Example 8

Express the following in their simplest forms;

(a) $\sqrt{8}$ (b) $\sqrt{20}$ (c) $\sqrt{90}$ (d) $\sqrt{18}$

Solution

(a) $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$
(b) $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$
(c) $\sqrt{90} = \sqrt{9 \times 10} = \sqrt{9} \times \sqrt{10} = 3\sqrt{2}$
(d) $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$

Example 9

Express the following as a single surd;

(a) $3\sqrt{2}$ (b) $4\sqrt{6}$ (c) $2\sqrt{3}$

Solution

(a) $3\sqrt{2} = \sqrt{3^2 \times 2} = \sqrt{9 \times 2} = \sqrt{18}$
(b) $4\sqrt{6} = \sqrt{4^2 \times 6} = \sqrt{16 \times 6} = \sqrt{96}$
(c) $2\sqrt{3} = \sqrt{2^2 \times 3} = \sqrt{4 \times 3} = \sqrt{12}$

Example 10

Evaluate the following;

(a) $(3 + \sqrt{2})(2 + \sqrt{3})$ (b) $(\sqrt{2} + \sqrt{3})^2$

Solution

(a) $(3 + \sqrt{2})(2 + \sqrt{3}) = 3(2 + \sqrt{3}) + \sqrt{2}(2 + \sqrt{3})$
$= 6 + 3\sqrt{3} + 2\sqrt{2} + \sqrt{6}$
(b) $(\sqrt{2} + \sqrt{3})^2 = (\sqrt{2} + \sqrt{3})(\sqrt{2} + \sqrt{3})$
$= \sqrt{2}(\sqrt{2} + \sqrt{3}) + \sqrt{3}(\sqrt{2} + \sqrt{3})$
$= 2 + \sqrt{6} + \sqrt{6} + 3$
$= 5 + 2\sqrt{6}$

Example 11

Rationalize the denominator of the following;

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{3-\sqrt{2}}$ (c) $\frac{2+\sqrt{3}}{2-\sqrt{3}}$ (d) $\frac{2\sqrt{2}+\sqrt{3}}{2\sqrt{2}-\sqrt{3}}$ (e) $\frac{1}{3-\sqrt{2}} - \frac{1}{3+\sqrt{2}}$ (f) $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$

Solution

(a) $\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$

$=\frac{\sqrt{2}}{2}$

(b) $\frac{1}{3-\sqrt{2}}=\frac{1}{3-\sqrt{2}}\left(\frac{3+\sqrt{2}}{3+\sqrt{2}}\right)$

$=\frac{3+\sqrt{2}}{3^2-(\sqrt{2})^2}$

$=\frac{3+\sqrt{2}}{9-2}$

$=\frac{3+\sqrt{2}}{7}$

(c) $\frac{2+\sqrt{3}}{2-\sqrt{3}}=\frac{2+\sqrt{3}}{2-\sqrt{3}}\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right)$

$=\frac{4+4\sqrt{3}+3}{4-3}$

$=\frac{7+4\sqrt{3}}{1}$

(d) $\frac{2\sqrt{2}+\sqrt{3}}{2\sqrt{2}-\sqrt{3}}=\frac{2\sqrt{2}+\sqrt{3}}{2\sqrt{2}-\sqrt{3}}\left(\frac{2\sqrt{2}+\sqrt{3}}{2\sqrt{2}+\sqrt{3}}\right)$

$=\frac{2\sqrt{2}(2\sqrt{2}+\sqrt{3})+\sqrt{3}(2\sqrt{2}+\sqrt{3})}{(2\sqrt{2})^2-(\sqrt{3})^2}$

$=\frac{8+2\sqrt{6}+2\sqrt{6}+3}{8-3}$

$=\frac{11+4\sqrt{6}}{5}$

(e) $\frac{1}{3-\sqrt{2}}-\frac{1}{3+\sqrt{2}}=\frac{1}{3-\sqrt{2}}\left(\frac{3+\sqrt{2}}{3+\sqrt{2}}\right)-\frac{1}{3+\sqrt{2}}\left(\frac{3-\sqrt{2}}{3+\sqrt{2}}\right)$

$=\frac{3+\sqrt{2}}{9-2}-\frac{3-\sqrt{2}}{9-2}$

$=\frac{3+\sqrt{2}-3+\sqrt{2}}{7}$

$=\frac{2\sqrt{2}}{7}$

(f) $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}=\frac{\sqrt{2}}{\sqrt{2}+(\sqrt{3}-\sqrt{5})}$ let $p=\sqrt{3}-\sqrt{5}$

$=\frac{\sqrt{2}(\sqrt{2}-p)}{(\sqrt{2}+p)(\sqrt{2}-p)}$

$=\frac{2-p\sqrt{2}}{2-p^2}$

$=\frac{2-(\sqrt{3}-\sqrt{5})\sqrt{2}}{2-(\sqrt{3}-\sqrt{5})^2}$

$=\frac{2-\sqrt{6}+\sqrt{10}}{2-(3-2\sqrt{6}+5)}$

$=\frac{2-\sqrt{6}+\sqrt{10}}{-6+2\sqrt{6}}$

$=\frac{(2-\sqrt{6}+\sqrt{10})(-6-2\sqrt{6})}{(-6+2\sqrt{6})(-6-2\sqrt{6})}$

$=\frac{-12+6\sqrt{6}-6\sqrt{10}-4\sqrt{6}+12-2\sqrt{60}}{36-24}$

$= \frac{2\sqrt{6}-6\sqrt{10}-2\sqrt{60}}{12}$

$= \frac{\sqrt{6}-3\sqrt{10}-\sqrt{60}}{6}$

Exercise 1.2

1. Simplify the given expressions;

(a) $\sqrt{54}$ (b) $\sqrt{72}$ (c) $\sqrt{68}$ (d) $\sqrt{84} \times \sqrt{140} \div \sqrt{120}$

2. Express as a single square root.

(a) $5\sqrt{5}$ (b) $x\sqrt{y}$ (c) $2a^2\sqrt{b}$

3. Rationalize the denominator of the following fractions, simplify your answer.

(a) $\frac{10}{\sqrt{5}}$ (b) $\frac{3+\sqrt{28}}{\sqrt{7}}$ (c) $\frac{\sqrt{5}+2}{\sqrt{5}-1}$ (d) $\frac{1+2\sqrt{2}}{5-3\sqrt{2}}$ (e) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

4. Express $\frac{1+\sqrt{3}}{(\sqrt{3}-1)^3}$ in the form $a + b\sqrt{c}$ where a, b and c are real roots.

5. Given that $\sqrt{3} = 1.732$ and$\sqrt{2} = 1.414$, evaluate the following;

(a) $\frac{1}{\sqrt{3}-\sqrt{2}}$ (b) $\left(3+\sqrt{2}\right)^2$ (c) $\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}}$ (d) $\sqrt{0.0675}$ to 3 significant figures.

6. Simplify $\frac{1+\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt{3}}$

7. Express in the form $a + b\sqrt{c}$ the following;

(a) $\frac{2}{\sqrt{2}+\sqrt{3}} + \frac{2}{\sqrt{2}-\sqrt{3}}$

(b) $\frac{2-\sqrt{3}}{2+\sqrt{3}} + \frac{4-\sqrt{3}}{2-\sqrt{3}}$

8. Simplify $\frac{(2+\sqrt{2})(3+\sqrt{5})(\sqrt{5}-2)}{(\sqrt{5}-1)(1+\sqrt{2})}$

1.3. Logarithms

- The logarithm of a positive number N to base a is defined as the power of a which is equal to N. Thus if

  $a^x = N$ ……………………………………..(i)

  Then x is the logarithm of N to the base a, written as

  $x = \log_a N$ ……………………………………(ii)

  Putting equation (ii) into equation (i)

  $\Longrightarrow \boldsymbol{a^{\log_a N} = N}$ …………..(*) this is so useful.
- Since we have $a^1 = a$ and $a^0 = 1$ it follows that

  $\log_a a = 1$and$\log_a 1 = 0$ for all $a\ (\neq 0)$
- Logarithm to base e is called natural logarithm and is denoted as $\ln x$ or $\log_e x$
- The laws for the manipulation of logarithms are derived directly from the laws of indices:

1. $\log_a bc = \log_a b + \log_a c$

   Proof

   Let $\log_a b = x$ and $\log_a c = y$

$\Rightarrow b = a^x$and$c = a^y$
$\Rightarrow bc = a^x.a^y$
$\Rightarrow bc = a^{x+y}$
Taking logarithm to base a of both sides
$\Rightarrow \log_a(bc) = \log_a a^{(x+y)}$
$\Rightarrow \log_a(bc) = x + y$
$\therefore \log_a bc = \log_a b + \log_a c$…………………■

2. $\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$
   Let $\log_a b = x$ and $\log_a c = y$
   $\Rightarrow b = a^x$and$c = a^y$
   $\Rightarrow \frac{b}{c} = \frac{a^x}{a^y}$
   $\Rightarrow \frac{b}{c} = a^{x-y}$
   Taking logarithm to base a of both sides
   $\Rightarrow \log_a\left(\frac{b}{c}\right) = \log_a a^{(x-y)}$
   $\Rightarrow \log_a\left(\frac{b}{c}\right) = x - y$
   $\therefore \log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$…………………■
3. $\log_a b^p = p\log_a b$
   Proof
   $\Rightarrow b^p = (a^x)^p = a^{px}$
   $\Rightarrow \log_a b^p = px = p\log_a b$
4. The change of base of a logarithm;
   $\log_a b = \frac{\log_c b}{\log_c a}$
   Proof
   Let $\log_a b = y$
   $\Rightarrow a^y = b$
   Taking logarithm to base c of both sides
   $\Rightarrow \log_c a^y = \log_c b$
   $\Rightarrow y\log_c a = \log_c b$
   $\Rightarrow y = \frac{\log_c b}{\log_c a}$
   $\therefore \log_a b = \frac{\log_c b}{\log_c a}$ ………………………■
5. $\log_a b = \frac{1}{\log_b a}$
   Proof
   Let $y = \log_a b$
   $\Rightarrow a^y = b$
   Taking logarithm to base b of both sides
   $\Rightarrow \log_b a^y = \log_b b$

$\Longrightarrow y\log_b a = 1$

$\Longrightarrow y = \frac{1}{\log_b a}$

$\therefore \log_a b = \frac{1}{\log_b a}$ ………………….■

Example 12

Evaluate;

(a) $\log_3 9$ (b) $\log_9 3$

Solution

(a) $\log_3 9 = \log_3 3^2 = 2$

(b) $\log_9 3 = \frac{1}{\log_3 9} = \frac{1}{\log_3 3^2} = \frac{1}{2}$

Example 13

Calculate $\log 5$ and $\log 0.125$ given that $\log 2 = 0.3010$

Solution

$$\log 5 = \log\left(\frac{10}{2}\right)$$
$$= \log 10 - \log 2$$
$$= 1 - 0.3010$$
$$= 0.699$$

$$\log 0.125 = \log\left(\frac{1}{8}\right)$$
$$= \log 1 - \log 8$$
$$= -3\log 2$$
$$= -3(0.3010)$$
$$= -0.903$$

Example 14

Express $\frac{1}{3}\log 8 - \log\left(\frac{2}{5}\right)$ as a single logarithm.

Solution

$$\frac{1}{3}\log 8 - \log\left(\frac{2}{5}\right) = \log 8^{1/3} - \log\left(\frac{2}{5}\right)$$
$$= \log\left(2 \div \left(\frac{2}{5}\right)\right)$$
$$= \log\left(2 \times \frac{5}{2}\right)$$
$$= \log 5$$

Example 15

Solve the following equations;

(a) $3^x = 10$ (b) $5^{4x-1} = 7^{x+2}$ (c) $2^{2x} - 2^x = 6$

Solution

(a) $3^x = 10$
Taking logarithm of both sides gives
$\log 3^x = \log 10$
$\Longrightarrow x = \frac{\log 10}{\log 3}$
$= \frac{1}{0.4771} = 2.0959$
(b) $5^{4x-1} = 7^{x+2}$
Taking logarithm of both sides gives
$\log 5^{4x-1} = \log 7^{x+2}$
$\Longrightarrow (4x-1)\log 5 = (x+2)\log 7$
$\Longrightarrow 4x\log 5 - \log 5 = x\log 7 + 2\log 7$
$\Longrightarrow x(4\log 5 - \log 7) = 2\log 7 + \log 5$
$\Longrightarrow x = \frac{2\log 7 + \log 5}{4\log 5 - \log 7}$
$\therefore x = 1.2247$
(c) $\Longrightarrow 2^{2x} - 2^x - 6 = 0$
$\Longrightarrow (2^x)^2 - (2^x) - 6 = 0$
Let $y = 2^x$
$\Longrightarrow y^2 - y - 6 = 0$
$\Longrightarrow (y+2)(y-3) = 0$
Solving gives $= -2$; $2^x = -2$ NA
$y = 3$; $2^x = 3$
Taking logarithms gives
$\log 2^x = \log 3$
$\Longrightarrow x = \frac{\log 3}{\log 2}$
$\therefore x = 1.5850$

Example 16

Solve the pair of simultaneous equations

$\log(y-x) = 0$and$2\log y = \log(21+x)$
Solution
If $\log(y-x) = 0$, then $y - x = 1$ using properties of logarithms,
$2\log y = \log(21+x)$ becomes
$\log y^2 = \log(21+x)$
$\Longrightarrow y^2 = (21+x)$

$\Rightarrow y^2 - x = 21$
We can now solve
$y - x = 1$ ……………………..(1)
$y^2 - x = 21$ …………………...(2)
Subtracting (2) from (1)
$\Rightarrow y^2 - y - 20 = 0$
$\Rightarrow (y-5)(y+4) = 0$
Solving gives $y = 5$ and $y = -4$
$\Rightarrow x = 5 - 1 = 4andx = -4 - 1 = -5$
∴The solutions are $x = 4, y = 5$ or $x = -5, y = -4.$

Example 17

Prove that $2\log_c(a+b) = 2\log_c a + \log_c\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$

Solution
From LHS
$\Rightarrow 2\log_c(a+b) = \log_c(a+b)^2$
$= \log_c(a^2 + 2ab + b^2)$
$= \log_c a^2\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$
$= \log_c a^2 + \log_c\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$
$= 2\log_c a + \log_c\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$
$= RHS$

$\therefore 2\log_c(a+b) = 2\log_c a + \log_c\left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right) \ldots\ldots\ldots\ldots\ldots. \blacksquare$

Example 18

Solve for x, $\log_x 9 + \log_{x^2} 3 = 2.5$

Solution
$\Rightarrow \log_x 9 + \frac{\log_x 3}{\log_x x^2} = 2.5$
$\Rightarrow \log_x 9 + \frac{1}{2}\log_x 3 = 2.5$
$\Rightarrow \log_x 9 + \log_x \sqrt{3} = 2.5$
$\Rightarrow \log_x 9\sqrt{3} = 2.5$
$\Rightarrow 9\sqrt{3} = x^{2.5}$
$\Rightarrow 3^{\left(2+\frac{1}{2}\right)} = x^{2.5}$
$\Rightarrow 3^{2.5} = x^{2.5}$
$\therefore x = 3$

Example 19

Solve the equation $2\log_4 x + 1 - \log_x 4 = 0$

Solution

$\Rightarrow 2\log_4 x + 1 - \frac{1}{\log_4 x} = 0$

Let $y = \log_4 x$

$\Rightarrow 2y + 1 - \frac{1}{y} = 0$

$\Rightarrow 2y^2 + y - 1 = 0$

$\Rightarrow (2y-1)(y+1) = 0$

$\Rightarrow y = 1/2 \text{or} y = -1$

$\Rightarrow \log_4 x = 1/2\ ; x = 4^{1/2} = 2$

$\Rightarrow \log_4 x = -1; x = 4^{-1} = 1/4$

$\therefore$the solutions are $x = 2$ and $x = 1/4$

Example 20

Solve the equation $\log_{10} x = \log_5 2x$

Solution

Let $y = \log_{10} x; 10^y = x$........(1)

Also $y = \log_5 2x$

$\Rightarrow 5^y = 2x$ ....(2)

Equation (1) ÷(2)

$\Rightarrow \frac{10^y}{5^y} = \frac{x}{2x}$

$\Rightarrow \frac{5^y.2^y}{5^y} = \frac{1}{2}$

$\Rightarrow 2^y = 2^{-1}$

$\Rightarrow y = -1$

$\Rightarrow x = 10^{-1}$

$\therefore x = \frac{1}{10}$

Exercise 1.3

1. Show that $\log_a(a^2 - x^2) = 2 + \log_a\left(1 - \frac{x^2}{a^2}\right)$
2. Show that $\log_a b\,.\log_b c\,.\log_c a = 1$
3. Solve the equation $3^{x^2} = 9^{x+4}$ $\quad Ans(x = -2, 4)$
4. Solve the equation $2^{3x+1} = 5^{x+1}$ $\quad Ans(x = 1.95)$
5. Solve the equation $9^x - 4 \times 3^x + 3 = 0$ $\quad Ans(x = 0, 1)$
6. Solve simultaneously the equations $2^{x+y} = 8, 3^{2x-y} = 27$ $\quad Ans(x = 2, y = 1)$
7. Solve the following equations;

(a) $5^{2x} - 5^{1+x} + 6 = 0$

(b) $\log_x 8 - \log_{x^2} 16 = 1$
(c) $\log_x 3 + \log_3 x = 2.5$
(d) $\log_{10}\left(\frac{x^2+24}{x}\right) = 1$
(e) $27^{x-3} = 3 \times 9^{x-2}$
(f) $2^x \times 3^{x+1} = 5^{2x+1}$
(g) $\log_{10}(x^2+9) - 2\log_{10} x = 1$

8. If $a^2 + b^2 = 23ab$, show hat $\log a + \log b = 2\log\left(\frac{a+b}{5}\right)$
9. Without using tables, show that $\frac{\log\sqrt{27}+\log\sqrt{8}-\log\sqrt{125}}{\log 6-\log 5} = \frac{3}{2}$
10. Express as a single logarithm in its simplest form
    $\log 2 + 2\log 18 - \frac{3}{2}\log 36$
11. Simplify the following expressions

(a) $3\log_2\left(\frac{5}{3}\right) - 2\log_2\left(\frac{10}{9}\right) + \log_2\left(\frac{1}{30}\right)$
(b) $\log_a\sqrt{(a^4+1)} - \frac{1}{2}\log_a\left(1+\frac{1}{a^4}\right)$
(c) $\log_4\{(a^2+1)^2 - (a^2-1)^2\} - \log_2 2a$

12. Find x from the equation

(a) $\log(x^2+2x) = 0.9031$
(b) $3^x - 3^{-x} = 6.832$
(c) $2^{2x+8} - 32(2^x) + 1 = 0$s
(d) $(2.4)^x = 0.59$

13. If $2\log_8 N = p, \log_2 2N = q, q - p = 4$, find N.
14. If $a^2 + b^2 = 7ab$, prove that $\log\frac{1}{3}(a+b) = \frac{1}{2}(\log a + \log b)$
15. If $x = \log_a bc\,, y = \log_b ca\,, z = \log_c ab$, prove that $x + y + z = xyz - 2$
16. Solve the equations

(a) $\log_{14} 4x = \log_7 x$
(b) $\log_n 4 + \log_4 n^2 = 3$

17. Determine $\log_x 2401$ when $x^3 = 7$
18. Prove that $\log x = -\frac{1}{2}\log\{1 - \{1 - \{1 - x^2\}^{-1}\}^{-1}\}$
19. Solve the following equations

(a) $\log_2 4x = 8\log_x 2$
(b) $\log_9 x = \log_3 3x$

20. Given that x and y are both positive, solve the simultaneous equations
    $\log(xy) = 7, \quad \log\left(\frac{x}{y}\right) = 1$
21. Solve the simultaneous equations;
    $2^x + 3^y = 5, \;\; 2^{x+3} - 3^{y+2} = 23$
22. Given that $\log(p - q + 1) = 0$and $\log(pq) + 1 = 0$, show that $p = q = \frac{1}{\sqrt{10}}$.
23. Solve the pair of simultaneous equations;

(a) $\log(x+y)=0,\ 2\log x=\log(y-1)$
(b) $\log_2 x^2+\log_2 y^3=1,\ \log_2 x-\log_2 y^3=4.$
24. Given that $\log_5 x=t$, express in terms of t;
(a) $\log_5 5x^2$ (b) $\log_x 5$ (c) $\log_{25} x$ (d) $\log_x 0.2$
25. Solve the equations;
(a) $\log_2 x^4+\log_2 4x=12$
(b) $\log_3 x=4\log_x 3$
(c) $3\log_8 x=2\log_x 8+5$
(d) $\log_5 x+\log_5(1/x^3)=2$
(e) $2\log_4 x+3\log_x 4=7$
(f) $\log_5 x+\log_x 25=3$

CHAPTER 2

EQUATIONS

2.1 Linear and simultaneous equations

Linear equations in one variable

- A linear equation is one which can be expressed in the form $ax+b=0$, e.g. $x+3=0$.

Example 21

Solve the following equations;

(a) $3x-7=x+3$
(b) $4(x+1)-3(x-5)=17$
(c) $\frac{x+5}{2}=\frac{3x+11}{5}$
(d) $\frac{2}{x+1}=\frac{3}{5}$
(e) $2x+\frac{x+7}{3}=\frac{4x-19}{5}$
(f) $\frac{1}{3}(5x-4)+\frac{1}{7}(x+2)=13-x$
(g) $\frac{6x+1}{2x-5}=\frac{3x-2}{x+1}$

Solution

(a) $\Rightarrow 3x-x=3+7$
$\Rightarrow 2x=10$
$\therefore x=5$
(b) $4x+4-3x+15=17$
$\Rightarrow x+19=17$
$\Rightarrow x=17-19$
$\therefore x=-2$

(c) Multiplying through by LCM=10

$\Longrightarrow 10\left(\frac{x+5}{2}\right) = 10\left(\frac{3x+11}{5}\right)$

$\Longrightarrow 5(x+5) = 2(3x+11)$

$\Longrightarrow 5x + 25 = 6x + 22$

$\Longrightarrow 5x - 6x = 22 - 25$

$\Longrightarrow -x = -3$

$\therefore x = 3$

(d) Cross-multiplying

$\Longrightarrow 10 = 3(x+1)$

$\Longrightarrow 10 = 3x + 3$

$\Longrightarrow 3x = 7$

$\therefore x = \frac{7}{3}$

(e) Multiplying through by the LCM=15

$\Longrightarrow 15(2x) + 15\left(\frac{x+7}{3}\right) = 15\left(\frac{4x-19}{5}\right)$

$\Longrightarrow 30x + 5(x+7) = 3(4x-19)$

$\Longrightarrow 30x + 5x + 35 = 12x - 57$

$\Longrightarrow 23x = -92$

$\therefore x = -4$

(f) Multiplying through by LCM =21

$\Longrightarrow (21)\frac{1}{3}(5x-4) + (21)\frac{1}{7}(x+2) = 21(13-x)$

$\Longrightarrow 7(5x-4) + 3(x+2) = 21(13-x)$

$\Longrightarrow 35x - 28 + 3x + 6 = 273 - 21x$

$\Longrightarrow 59x = 295$

$\therefore x = 5$

(g) Cross-multiplying

$\Longrightarrow (6x+1)(x+1) = (3x-2)(2x-5)$

$\Longrightarrow 6x^2 + 7x + 1 = 6x^2 - 19x + 10$

$\Longrightarrow 26x = 9$

$\therefore x = \frac{9}{26}$

Linear Simultaneous equations in two variables

- There are many methods of solving these equations and amongst which we shall have;

  (i) Elimination method

  (ii) Substitution

Method 1: Elimination.

Example 22

Solve the pair of simultaneous equations;

$5x - 7y = 27$
$2x + 3y = 5$

Solution

$5x - 7y = 27$ ………………..(1)
$2x + 3y = 5$ ………………….(2)
Equation (1)x3 and Equation (2) x7 in order to eliminate y
$15x - 21y = 81$ …………………(3)
$14x + 12 = 35$ …………………..(4)
Adding (3) and (4) give
$29x = 116$
$x = 4$
Substituting $x = 4$ into equation (2) to find y
$\Longrightarrow 2(4) + 3y = 5$
$\Longrightarrow 8 + 3y = 5$
$y = -1$
$\therefore$the solution is $x = 4, y = -1$
Method II: Substitution

Example 23
Solve the simultaneous equations
$3x + 4y - 27 = 0$
$5x + y - 11 = 0$
Solution
$3x + 4y - 27 = 0$ ……………..(1)
$5x + y - 11 = 0$ ………………..(2)
Making y the subject in equation (2) gives
$y = 11 - 5x$ …………………………(3)
Substituting (3) into (1) gives
$3x + 4(11 - 5x) - 27 = 0$
$\Longrightarrow 3x + 44 - 20x - 27 = 0$

$\Rightarrow -17x = -17$
$\therefore x = 1$
Substituting $x = 1$ into (3) gives
$y = 11 - 5(1)$
$\therefore y = 6$
$\underline{\therefore \text{the solution is } x = 1, y = 6}$

Linear simultaneous equations in three variables

There are a variety of methods for solving these equations

Method I: Eliminations method.

Example 24

Solve the simultaneous equations;

$2x + 3y + 4z = 8,\ \ 3x - 2y - 3z = -2,\ \ 5x + 4y + 2z = 3$

Solution

$2x + 3y + 4z = 8$ ……………………..(1)
$3x - 2y - 3z = -2$ …………………..(2)
$5x + 4y + 2z = 3$ ……………………..(3)
$y$and$z$ may be eliminated easily, we eliminate y as follows
$2 \times (2) + (3)$: $11x - 4z = -1$………………(4)
$3 \times (3) - 4(1)$: $7x - 10 = -23$………………(5)
$5 \times (4) - 2 \times (5)$: $41x = 41$
$\therefore x = 1, z = 3, y = -2$

Method 2: Row reduction to echelon form.

This involves reducing the matrix of coefficients into a meaning matrix of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

Example 25

Solve the simultaneous equations;

$2x + 3y + 4z = 8,\ \ 3x - 2y - 3z = -2,\ \ 5x + 4y + 2z = 3$

Solution

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 2 & 3 & 4 \\ 3 & -2 & -3 \\ 5 & 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ 3 \end{pmatrix}$$

$2R_2 - 3R_1 \longrightarrow R_2; 2R_3 - 5R_1 \longrightarrow R_3$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 2 & 3 & 4 \\ 0 & -13 & -18 \\ 0 & -7 & -16 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -28 \\ -34 \end{pmatrix}$$

$13R_3 - 7R_2 \longrightarrow R_3$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 2 & 3 & 4 \\ 0 & -13 & -18 \\ 0 & 0 & -82 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -28 \\ -246 \end{pmatrix}$$

$\frac{R_1}{2} \longrightarrow R_1, \frac{R_2}{-13} \longrightarrow R_2, \frac{R_3}{-82} \longrightarrow R_3$

$$\begin{pmatrix} 1 & 1.5 & 2 \\ 0 & 1 & \frac{18}{13} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{28}{13} \\ 3 \end{pmatrix}$$

$\Rightarrow z = 3$

$\Rightarrow y + \frac{18}{13}z = \frac{28}{13};\ \ y = \frac{28}{13} - \frac{18}{13}(3) = -2$

$\Rightarrow x + 1.5y + 2z = 4$

$\Rightarrow x = 4 - (1.5 \times -2) - (2 \times 3) = 1$

$\therefore \underline{x = 1, y = -2, z = 3}$

Method 3: Substitution method

This involves making one letter the subject and then we substitutte it in the other two equations

Example 26

Solve the following equations simultaneously

$2a + b + 3c = 11$

$a + 2b - 2c = 3$

$4a + 3b + c = 15$

Solution

$2a + b + 3c = 11$ ....(1)

$a + 2b - 2c = 3$ ....(2)

$4a + 3b + c = 15$ ....(3)

From (2) $a = 3 - 2b + 2c$................(4)

Putting equation (4) into (1) and (3) gives

$2(3 - 2b + 2c) + b + 3c = 11$

$\Rightarrow -3b + 7c = 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5)$

$4(3 - 2b + 2c) + 3b + c = 15$

$\Rightarrow -5b + 9c = 3$ ....(6)

Solving (5) and (6) simultaneously

From (5) $b = \frac{5-7c}{-3}$....................(7)

Putting (7) into (6) gives

$-5\left(\frac{5-7c}{-3}\right) + 9c = 3$

$\Longrightarrow 25 - 35c + 27c = 9$

$\Longrightarrow -8c = -16;\ \ c = 2$

From (7) $b = \frac{5-7c}{-3} = \frac{5-7(2)}{-3} = 3$

From (4); $a = 3 - 2b + 2c = 3 - 2(3) + 2(2) = 1$

$\underline{\therefore a = 1, b = 3, c = 2}$

Exercise 2.1

1. Solve the following equations;

(a) $\frac{x+1}{3} - \frac{x-2}{4} = \frac{2x+3}{6}$

(b) $2x + 3(x-1) = 4x + 12 Ans(x = 15)$

(c) $\frac{x+5}{5} = \frac{x-1}{6} Ans(x = -35)$

2. Solve the following equations;

(a) $3(5-x) - 4(3x-2) = 27$

(b) $\frac{6}{x+8} = \frac{5}{3x+4}$

(c) $5x - \frac{3x+1}{2} = \frac{7x+9}{3}$

(d) $\frac{x-4}{6} - 2x + 1 = \frac{3x-4}{2}$

(e) $\frac{1}{5}(2x-3) - \frac{1}{4}(5x-2) = x + 1$

3. Solve the following equations simultaneously;

(a) $2x - 3y = 7,\ 2x + 3y = 1 Ans(2, -1)$

(b) $3x - y = 1, 5x + y = 7 Ans(1, 2)$

(c) $2x - 7y = 1,\ 2x + 3y = 11 Ans(4, 1)$

(d) $3x - 4y = 5,\ 6x - 4y = 2 Ans(-1, -2)$

4. Solve the following equations simultaneously

(a) $3p + 2q + 5r = 7, 2p - 4q + 9r = 9,\ 6p - 8q + 3r = 4$

(b) $2x + 3y + 4z = -4, 4x + 2y + 3z = -11, 3x + 4y + 2z = -3$

(c) $d - 2e + 3f = 4, 5d + 6e - 7f = 8, 7d - 5e + 6f = 4$

2.2 Quadratic equations:

- An equation of the form $ax^2 + bx + c = 0$; $(a \neq 0)$ is called a quadratic equation.
- There are three types of quadratic equations;

Type I; $\boldsymbol{ax^2 + c = 0}$

Example 27

Solve the following equations

(a) $x^2 - 16 = 0$ (b) $x^2 - 24 = 0$

Solution

(a) $\Rightarrow x^2 = 16$

$\Rightarrow x^2 = \pm\sqrt{16}$

$\therefore x = \pm 4$

(b) $\Rightarrow 2x^2 = 24$

$\Rightarrow x^2 = 6$

$\Rightarrow x = \pm\sqrt{6}$

$\therefore x = \pm 2.4495$

Type II: $ax^2 + bx = 0$

Example 28

Solve the equations;

(a) $x^2 - 7x = 0$ (b) $3x^2 + 5x = 0$

Solution

(a) $\Rightarrow x(x - 7) = 0$

$\Rightarrow x = 0 \; or \; x - 7 = 0$

$\therefore x = 0 \; or \; x = 7$

$\therefore$the solutions are $x = 0$ and $x = 7$

(b) $\Rightarrow x(3x + 5) = 0$

$\Rightarrow x = 0 \; or \; 3x + 5 = 0$

$\therefore x = 0 \; or \; x = -5/3$

$\therefore$the solutions are $x = 0$ and $x = -5/3$.

Type III: $ax^2 + bx + c = 0$:

There are basically three methods of solving these equations;

Method I: Factorization

Example 29

Solve the following equations;

(a) $x^2 + 5x + 6 = 0$

(b) $2x^2 - 13x - 24 = 0$

(c) $x^2 - x - 10 = x + 5$

(d) $(3x + 1)(2x - 1) - (x + 2)^2 = 5$

(e) $\frac{4}{x-1} + \frac{3}{x} = 3$

(f) $\frac{2x+1}{x+5} = \frac{3x-1}{x+7}$

Solution

(a) $\Longrightarrow (x+2)(x+3) = 0$
$\Longrightarrow x+2 = 0 \ or \ x+3 = 0$
$\Longrightarrow x = -2 \ or \ x = -3$
∴the solutions are $x = -2$ and $x = -3$

(b) $\Longrightarrow (2x+3)(x-8) = 0$
$\Longrightarrow 2x+3 = 0 \ or \ x-8 = 0$
$\Longrightarrow x = -3/2 \ or \ x = 8$
∴the solutions are $x = -3/2$ and $x = 8$

(c) $\Longrightarrow x^2 - 2x - 15 = 0$
$\Longrightarrow (x+3)(x-5) = 0$
$\Longrightarrow x+3 = 0 \ or \ x-5 = 0$
$\Longrightarrow x = -3 \ \ or \ x = 5$
∴the solutions are $x = -3$ and $x = 5$

(d) $\Longrightarrow 6x^2 - x - 1 - (x^2 + 4x + 4) = 5$
$\Longrightarrow 5x^2 - 5x - 5 = 5$
$\Longrightarrow 5x^2 - 5x - 10 = 0$
$\Longrightarrow 5(x^2 - x - 2) = 0$
$\Longrightarrow 5(x+1)(x-2) = 0$
$\Longrightarrow x+1 = 0 \ or \ x-2 = 0$
$\Longrightarrow x = -1 \ or \ x = 2$
∴the solutions are $x = -1$ and $x = 2$

(e) Multiplying through by the $LCM = x(x-1)$

$\Longrightarrow x(x-1)\left(\frac{4}{x-1}\right) + x(x-1)\left(\frac{3}{x}\right) = 3x(x-1)$

$\Longrightarrow 4x + 3(x-1) = 3x(x-1)$

$\Longrightarrow 4x + 3x - 3 = 3x^2 - 3x$

$\Longrightarrow 3x^2 - 10x + 3 = 0$

$\Longrightarrow (3x-1)(x-3) = 0$

$\Longrightarrow 3x - 1 = 0 \ or \ x - 3 = 0$

$\Longrightarrow x = 1/3 \text{or} x = 3$

∴the solutions are $x = 1/3$ and $x = 3$

(f) Cross-multiply
$\Longrightarrow (2x+1)(x+7) = (3x-1)(x+5)$

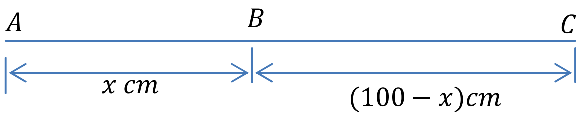
$\Longrightarrow 2x^2 + 15x + 7 = 3x^2 + 14x - 5$
$\Longrightarrow x^2 - x - 12 = 0$
$\Longrightarrow (x + 3)(x - 4) = 0$
$\Longrightarrow x + 3 = 0 \text{ or } x - 4 = 0$
$\Longrightarrow x = -3 \text{ or } x = 4$
$\therefore$the solutions are $x = -3$ and $x = 4$

Example 30

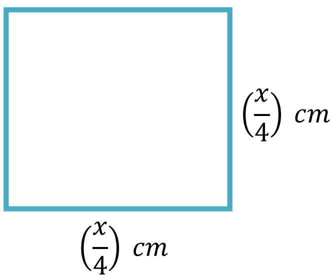
A piece of wire of length 1m is cut into two parts and each part is bent to form a square. If the total area of the two squares formed is $325cm^2$, find the perimeter of each square.

Solution

Let one of the pieces of wire be of length $x\ cm$. Then the other piece is of length $(100 - x)cm$.



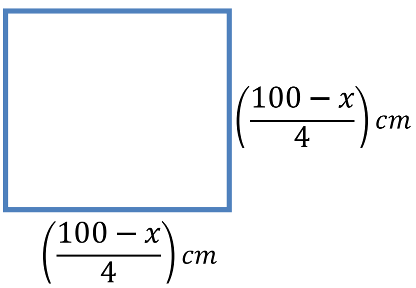
The square formed from the piece AB has sides of length $\frac{x}{4}\ cm$



The area, A_1, of this square is given by;

$$A_1 = \left(\frac{x}{4}\right)\left(\frac{x}{4}\right) = \left(\frac{x^2}{16}\right) cm^2$$

The square formed from the piece BC has sides of length $\left(\frac{100-x}{4}\right) cm$



The area, A_2, of this square is given by

$$A_2 = \left(\frac{100-x}{4}\right)\left(\frac{100-x}{4}\right) = \left(\frac{(100-x)^2}{16}\right) cm^2$$

Since the total area of the two squares is 325 cm^2

$\Longrightarrow A_1 + A_2 = 325$

$\Longrightarrow \left(\frac{x^2}{16}\right) + \left(\frac{(100-x)^2}{16}\right) = 325$

$\Longrightarrow x^2 + (100 - x)^2 = 5200$

$\Longrightarrow x^2 + 10000 - 200x + x^2 = 5200$

$\Longrightarrow 2x^2 - 200x + 4800 = 0$

$\Longrightarrow 2(x^2 - 100x + 2400) = 0$

$\Longrightarrow 2(x - 40)(x - 60) = 0$

$\Longrightarrow x - 40 = 0 \ or \ x - 60 = 0$

$\therefore x = 40 \ or \ x = 60$

If $x = 40\ cm$, the square formed from the piece of wire AB has perimeter 40cm, and the square formed from the piece of wire BC has perimeter 60 cm.

If $x = 60\ cm$, the square formed from the piece of wire AB has perimeter 60cm, and the square formed from the piece of wire BC has perimeter 40 cm.

Method 2: Completing squares:

- Completing squares is the process of adding half the coefficient of the x term squared.
- If we want to make $x^2 + bx$ into a perfect square we add $\left(\frac{b}{2}\right)^2$.

$x^2 + bx + \left(\frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$

$= \left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right)$

$= \left(x + \frac{b}{2}\right)^2$ which is a perfect square

- The process of completing squares is used to express a quadratic expression $ax^2 + bx + c$in the form$a(x + p)^2 + q$ where p and q are constants.
- First we look at those quadratic expressions in which $a = 1$.

- In general $x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$

$$= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$$

Example 31

Express $x^2 + 6x - 1$ in the form $a(x + p)^2 + q$. Hence solve the equation

$x^2 + 6x - 1 = 0$.

Solution

$\Rightarrow x^2 + 6x - 1 = x^2 + 6x + 9 - 1 - 9$

$= (x + 3)^2 - 10$

Rewriting the equation $x^2 + 6x - 1 = 0$gives

$(x + 3)^2 - 10 = 0$

$\Rightarrow (x + 3)^2 = 10$

$\Rightarrow x + 3 = \pm\sqrt{10}$

$\Rightarrow x = -3 \pm \sqrt{10}$

$\therefore$the solutions are $x = 0.1623$ and $x = -6.1623$

Qn. Use the method of completing squares to solve the equation $x^2 - 3x + 1 = 0$.

- We can now look at examples of the form $ax^2 + bx + c$.

Example 32

Express $2x^2 + 8x + 5$ in the form $a(x + p)^2 + q$ and state the values of a, p and q.

Solution

$\Rightarrow 2x^2 + 8x + 5 = 2\left(x^2 + 4x + \frac{5}{2}\right)$

$= 2\left(x^2 + 4x + 4 + \frac{5}{2} - 4\right)$

$= 2\left[(x + 2)^2 - \frac{3}{2}\right]$

$= 2(x + 2)^2 - 3$

Therefore, $a = 2, p = 2$ and $q = -3$

Example 33

Express $3x^2 + 15x + 20$ in the form $a(x + p)^2 + q$. Hence show that the equation $3x^2 + 15x + 20 = 0$ has no real roots.

Solution

$\Rightarrow 3x^2 + 15x + 20 = 3\left(x^2 + 5x + \frac{20}{3}\right)$

$= 3\left(x^2 + 5x + \left(\frac{5}{2}\right)^2 + \frac{20}{3} - \left(\frac{5}{2}\right)^2\right)$

$$= 3\left[\left(x+\frac{5}{2}\right)^2+\frac{20}{3}-\frac{25}{4}\right]$$
$$= 3\left[\left(x+\frac{5}{2}\right)^2+\frac{5}{12}\right]$$
$$= 3\left(x+\frac{5}{2}\right)^2+\frac{5}{4}$$

Rewriting the equation $3x^2+15x+20=0$ gives$3\left(x+\frac{5}{2}\right)^2+\frac{5}{4}=0$

$\Rightarrow 3\left(x+\frac{5}{2}\right)^2 = -\frac{5}{4}$

$\Rightarrow \left(x+\frac{5}{2}\right)^2 = -\frac{5}{12}$

$\therefore x+\frac{5}{2} = \pm\sqrt{-\frac{5}{12}}$

$\therefore$ Since $\sqrt{-\frac{5}{12}}$ is not real, the equation $3x^2+15x+20=0$ has no real roots.

Example 34

Find the minimum value of $f(x)=13+6x+3x^2$ and state the value of x for which this occurs.

Solution

$f(x) = 13+6x+3x^2$
$$= 3\left(x^2+2x+\frac{13}{3}\right)$$
$$= 3\left(x^2+2x+1+\frac{13}{3}-1\right)$$
$$= 3\left[(x+1)^2+\frac{10}{3}\right]$$
$$= 3(x+1)^2+10$$

The minimum value of $f(x)$ is 10 and it occurs when $(x+1)^2 = 0; x=-1$

Example 35

Find by completing squares, the greatest value of the function $f(x) = 1-6x-x^2$ and state the value of x for which it occurs.

Solution

$f(x) = 1-6x-x^2$
$$= 10-(9+6x+x^2)$$
$$= 10-(3+x)^2$$

Since$(3+x)^2$ is the square of a real number it cannot be negative, it is zero when x=-3, otherwise it is positive. Consequently $10-(3+x)^2$is always less than or equal to 10.

$\therefore$the greatest value of the function is 10 and this occurs when x=-3.

Example 36

Show that $3x^2 + 10x + 9$ cannot be negative and find its least value.

Solution
By completing squares;
$$\Rightarrow 3x^2 + 10x + 9 = 3\left(x^2 + \frac{10}{3}x + 3\right)$$
$$= 3\left(x^2 + \frac{10}{3}x + \frac{25}{9} + 3 - \frac{25}{9}\right)$$
$$= 3\left[\left(x + \frac{5}{3}\right)^2 + \frac{2}{9}\right]$$
$$= 3\left(x + \frac{5}{3}\right)^2 + \frac{2}{3}$$
Since $\left(x + \frac{5}{3}\right)^2$ is a square, the least value it can take is zero, so that $3x^2 + 10x + 9$ cannot be zero and its least value is $\frac{2}{3}$.

Example 37

Find the least value of the function $f(x) = \frac{1}{2-2x-x^2}$ and state the value of x for which it occurs.

Solution
$$f(x) = \frac{1}{2-2x-x^2}$$
$$= \frac{1}{-(x^2+2x-2)}$$
$$= \frac{1}{-(x^2+2x+1-2-1)}$$
$$= \frac{1}{-[(x+1)^2-3]}$$
$$= \frac{1}{3-(x+1)^2}$$
The least value of $f(x)$ is $1/3$ and it occurs when $(x + 1)^2 = 0$ $i.e.\ x = -1$

Example 38

Find the greatest value of $g(x) = \frac{1}{13+6x+3x^2}$ and state the values of x for which it occurs.

Solution
$$g(x) = \frac{1}{13+6x+3x^2}$$
$$= \frac{1}{3\left(x^2+2x+\frac{13}{3}\right)}$$
$$= \frac{1}{3\left(x^2+2x+1+\frac{13}{3}-1\right)}$$

$= \frac{1}{3\left[(x+1)^2 + \frac{10}{3}\right]}$
$= \frac{1}{3(x+1)^2 + 10}$
∴the greatest value of $g(x)$ is $1/10$ and it occurs when $(x+1)^2 = 0$ i.e. $x = -1$

Method 3: Quadratic formula:

- If $ax^2 + bx + c = 0$, where a, b and c are constants with $a \neq 0$, then

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

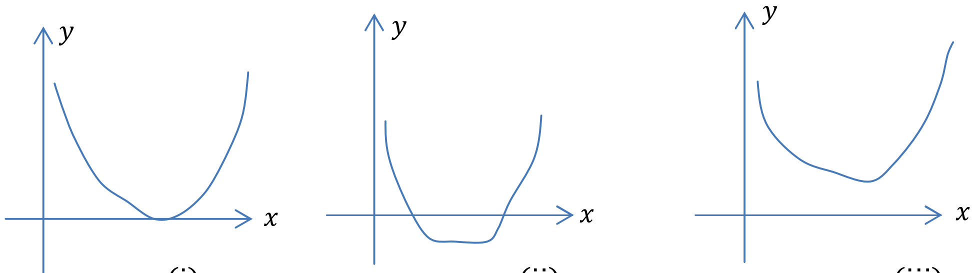
Example 39

Solve the equations;

(a) $x^2 + 11 = 7x$
(b) $2x^2 - 11x + 12 = 0$

Solution

(a) $x^2 - 7x + 11 = 0$
Using $a = 1, b = -7$ and $c = 11$
$\Rightarrow x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(11)}}{2(1)}$
$\therefore x = 4.618$ or $x = 2.382$
(b) $2x^2 - 11x + 12 = 0$
Using $a = 2, b = -11$ and $c = 12$
$\Rightarrow x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(2)(12)}}{2(2)}$
$\therefore x = 4$ or $x = 1.5$
Qn. Solve the equation;
(a) $x^2 - 8x + 4 = 0$ (b) $5x^2 + 4x + 10 = 0$
Discriminant of a quadratic equation:
- The quantity $D = b^2 - 4ac$ is called the discriminant of the quadratic equation $ax^2 + bx + c = 0$.
- The type of root which arises from a quadratic equation depends on the value of the discriminant.
If;
(i) $D = 0$; the roots of the equation are $x = \frac{-b}{2a}$ and $x = \frac{-b}{2a}$ and these are called repeated/equal roots.
(ii) $D > 0$; the roots are unequal/distinct and are $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$
(iii) $D < 0$; there are no real roots of the equation.
- The following are the graphs for the above conditions for $a > 0$



(i) (ii) (iii)

Note: If the roots are real then $D \geq 0$

Example 40

Find the discriminant and stat whether the equation will have two real roots, no root or a repeated root in each of the following cases.

(a) $3x^2 - 5x + 2 = 0$ (b) $4x^2 - 28x + 49 = 0$ (c) $5x^2 - 7x + 3 = 0$

Solution

(a) Calculating the discriminant with $a = 3, b = -5$ and $c = 2$ gives

$D = b^2 - 4ac$

$= (-5)^2 - 4(3)(2)$

$= 1$

Since $D > 0$, then the equation $3x^2 - 5x + 2 = 0$ has two real distinct roots.

(b) Calculating the discriminant with $a = 4, b = -28$ and $c = 49$ gives

$D = b^2 - 4ac$

$= (-28)^2 - 4(4)(49)$

$= 0$

Since $D = 0$, then the equation $4x^2 - 28x + 49 = 0$ has equal roots.

(c) Calculating the discriminant with $a = 5, b = -7$ and $c = 3$ gives

$D = b^2 - 4ac$

$= (-7)^2 - 4(5)(3)$

$= -11$

Since $D < 0$, then the equation $5x^2 - 7x + 3 = 0$ has no real roots.

Example 41

Find the values of the constant k given that the equation

$(5k + 1)x^2 - 8kx + 3k = 0$has a repeated root.

Solution

For repeated root if the discriminant $D = 0$

Calculating the discriminant of the equation with $a = 5k + 1, b = -8k$ and $c = 3k$

$\Rightarrow D = b^2 - 4ac$

$= (-8k)^2 - 4(5k+1)(3k)$
$= 64k^2 - 12k(5k+1)$
$\therefore D = 4k^2 - 12k$
Putting $D = 0$ and factorizing gives
$4k^2 - 12k = 0$
$4k(k-3) = 0$
$\therefore k = 0 \; or \; k - 3 = 0$
$\therefore k = 0 \; or \; k = 3$
$\therefore$the required values of k are 0 and 3

Example 42

Find the range of values k can take for $kx^2 - 4x + 5 - k = 0$ to have two real distinct roots.

Solution
The equation $kx^2 - 4x + 5 - k = 0$ has two real distinct roots if $D > 0$
Calculating the discriminant of the equation with $a = k, b = -4$ and $c = (5-k)$ gives;
$D = b^2 - 4ac$
$= (-4)^2 - 4(k)(5-k)$
$= 16 - 20k + 4k^2$
Putting $D > 0$
$\Rightarrow 4(k^2 - 5k + 4) > 0$
$\Rightarrow (k-4)(k-1) > 0$
Let $k - 4 = 0; k = 4$ and $k - 1 = 0; k = 1$
$\therefore k = 1, 4$
Investigation table

	$k < 1$	$1 < k < 4$	$k > 4$
$(k-1)$	-	+	+
$(k-4)$	-	-	+
$(k-1)(k-4)$	+	-	+

The required ranges are $k < 1$and $k > 4$.

Example 43

If x is real, show that the expression $y = (x^2 + x + 1)/(x+1)$ can have no real value between $-3$and 1.

Solution
Rearranging as a quadratic in x
$\Rightarrow (x+1)y = x^2 + x + 1$

$\Rightarrow x^2 + (1-y)x + 1 - y = 0$

For x to be real, $(1-y)^2 - 4(1)(1-y) \geq 0$

$\Rightarrow (1-y)(-3-y) \geq 0$

Let $1 - y = 0, y = 1$ *and* $(-3-y) = 0; y = -3$

$\therefore y = -3, 1$

Investigation table

	$y < -3$	$-3 < y < 1$	$y > 1$
$(1-y)$	+	+	-
$(-3-y)$	+	-	-
$(1-y)(-3-y)$	+	-	+

$\therefore$the expression has no real vales between -3 and 1.

Exercise 2.2a

1. Solve the following quadratic equations;

(a) $x^2 - 5x + 6 = 0$ (b) $x^2 - 3x - 4 = 0$ (c) $2x^2 - 3x - 5 = 0$

(d)$\frac{2}{x+1} = \frac{x}{3-2x}(e)\frac{x+2}{4} = \frac{x}{4-x}$ (f) $\frac{5}{x+3} + \frac{7}{x-1} = 8$

2. The perimeter of a rectangle is 34 cm. Given that the diagonal is of length 13 cm, and that the width is $x\ cm$, derive the equation $x^2 - 17x + 60 = 0$. Hence find the dimensions of the rectangle.
3. A train usually covers a journey of 240 km at a steady speed of $v\ kmh^{-1}$. One day, due to adverse weather conditions, it reduces its speed by $40kmh^{-1}$and the journey takes one hour longer. Derive the equation $v^2 - 40v - 9600 = 0$, and solve it to find the value of v.
4. Use the method of completing squares the solutions to ach of the following quadratic equations in the form $a \pm b\sqrt{n}$, where a and b are rational, and n is an integer .

(a) $x^2 - 4x + 1 = 0$ (b) $x^2 + 12x + 5 = 0$ (c) $x^2 - 9x + 10 = 0$

5. Use the method of completing squares the solutions to ach of the following quadratic equations in the form $a \pm b\sqrt{n}$, where a and b are rational, and n is an integer .

   (a) $2x^2 - 3x - 3 = 0$(b) $3x^2 + 5x - 1 = 0$ (c)$7x^2 - 14x + 5 = 0$

6. Express $x^2 + 4x + 7$ in the form $(x+p)^2 + q$. Hence show that the equation $x^2 + 4x + 7 = 0$hasno roots.
7. Find the real roots, if any, of the following quadratic equations, giving your answers in surd form;

(a) $x^2 - 3x + 1 = 0$ (b) $2x^2 + 3x - 1 = 0$ (c) $1 - 2x - x^2 = 0$

8. Find the values of k for which the following equations have equal roots

(a) $x^2 + 2kx + k + 6 = 0$ (b) $(x+1)(x+3) = k$

9. Find the range of values of p for which each of the given equations has two distinct real roots.

(a) $x^2 + 2px - 5p = 0$ (b) $3x^2 + 3px + p^2 = 1$ (c) $x(x+3) = p(x-1)$

(d) $p(x^2-1)=3x+2$

10. Find the values of p for which the expression $x^2+(p+3)x+2p+3$ is a perfect square.
11. Prove that the roots of the equation $(k+3)x^2+(6-2k)x+k-1=0$ are real if and only if k is not greater than $3/2$.
12. Show that the function $f(x)=\frac{x^2-x-6}{x-1}$ takes on all real values.
13. Show that the equation $kx(1-x)=1$ has no real roots if $0<k<4$.
14. Show that for all real values of y, the expression $\frac{y^2-2y-1}{y^2+y+2}$ always lies between $-4/7$ and 4.
15. Find the values of μ for which the equation $10x^2+4x+1=2\mu x(2-x)$ has equal roots.
16. Show that, if $y=\frac{x^2+1}{x^2-a^2}$, y takes all values twice, except those for which $-\frac{1}{a^2}\leq y\leq 1$.
17. Given that the equation $x^2-3bx+(4b+1)=0$ has a repeated root, find the possible values of the constant b.
18. Show that there is no real value off the constant c for which the equation $cx^2+(4c+1)x+(c+2)=0$ has a repeated root.

Disguised Quadratic equations:

- These are equations which don't appear to be quadratic but in actual sense they are.

Example 44

Solve the equation$x^4+5x^2-14=0$

Solution

$\Rightarrow (x^2)^2+5x^2-14=0$

Let $y=x^2$

$\Rightarrow (y)^2+5y-14=0$

$\Rightarrow (y+7)(y-2)=0$

$\therefore y+7=0\ or\ y-2=0$

$\therefore y=-7;\ x^2=-7\ NAory=2;\ x^2=2$

$\therefore x=\pm\sqrt{2}=\pm 1.4142$

Example 45

Solve the following equations;

(a) $x-9\sqrt{x}+20=0$

(b) $x^2+2x-4+\frac{3}{x^2+2x}=0$

(c) $x^{4/3}+16x^{-4/3}=17$

Solution

(a) Rewriting the equation gives;

$(\sqrt{x})^2 - 9(\sqrt{x}) + 20 = 0$

Let $y = \sqrt{x}$

$\Rightarrow y^2 - 9y + 20 = 0$

$\Rightarrow (y-4)(y-5) = 0$

$\Rightarrow y = 4;\ \sqrt{x} = 4, \therefore x = 16$

$\Rightarrow y = 5;\ \sqrt{x} = 5, x = 25$

$\therefore$the solutions are $x = 16$ and $x = 25$.

(b) With $z = x^2 + 2x$ gives

$z - 4 + \frac{3}{z} = 0$

$\Rightarrow z^2 - 4z + 3 = 0$

$\Rightarrow (z-3)(z-1) = 0$

$\Rightarrow z = 1\ or\ 3$

With $z = 3$

$\Rightarrow x^2 + 2x = 3$

$\Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow (x+3)(x-1) = 0$

$\Rightarrow x = -3\ or\ x = 1$

With $z = 1$

$\Rightarrow x^2 + 2x - 1 = 0$

$x = \frac{-2 \pm \sqrt{4-4(1)(-1)}}{2}$

$x = 0.4142$ or $x = -2.4142$

$\therefore$the solution are $1, 0.4142, -2.4142, -3$

(c) Rearranging the equation gives

$x^{4/3} + \frac{16}{x^{4/3}} = 17$

Let $y = x^{4/3}$

$\Rightarrow y + \frac{16}{y} = 17$

$\Rightarrow y^2 - 17y + 16 = 0$

$\Rightarrow (y-16)(y-1) = 0$

$\Rightarrow y = 16;\ x^{4/3} = 16\ ; x = (16)^{3/4} = 8$

$\Rightarrow y = 1; x = 1$

$\therefore$the solutions are $x = 1$ and $x = 8$

Sketching the graphs of the quadratic functions

Example 46

Express $y = x^2 - 2x - 8$ in the form $a(x + p) + q$ and hence sketch the graph of $y = x^2 - 2x - 8$.

Solution

Completing squares

$\Rightarrow y = x^2 - 2x - 8 = x^2 - 2x + 1 - 8 - 1$

$\quad = (x - 1)^2 - 9$

Since $(x - 1)^2 \geq 0$ with equality holding when $x = 1$, the minimum value of y is -9 and this occurs when $(x - 1)^2 = 0; x = 1$

The coordinates (1, -9) is the minimum point of the graph.

The graph cuts the $x - axis$ when $y = 0$ i.e.

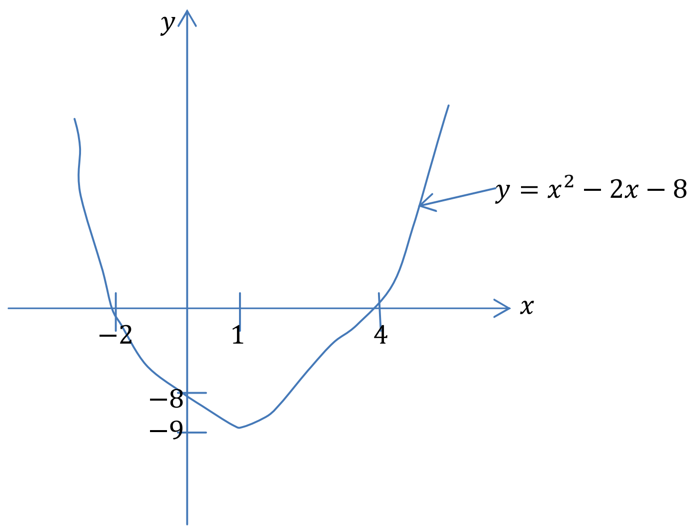
$x^2 - 2x - 8 = 0$

$(x + 2)(x - 4) = 0$

$x = -2\ or = 4$

(-2, 0), (4, 0)

The graph cuts the y-axis when $x = 0$. That is when $y = -8$; $(0, -8)$



Qn. Express $y = -x^2 + 10x - 21$ in the form $a(x + p)^2 + q$ and hence sketch the graph of $y = -x^2 + 10x - 21$

Exercise 2.2b

1. Solve the following equations for x;

(a) $x^4 - 13x^2 + 36 = 0$ (b) $8\sqrt{x} = 15 + x$ (c) $x^8 + 16 = 17x^4$

(d) $x^2 + 1 = \frac{6}{x^2}$ (e) $2 + \frac{10}{x} = \frac{9}{\sqrt{x}}$ (f) $\frac{5}{x^2} = x^2 + \frac{4}{x^6}$ (g) $x(x + 1) + \frac{24}{x(x+1)} = 14$

2. Solve the following equations;

(a) $(x+3)^2-5(x+3)+4=0$ (b) $(3x-1)^2+6(3x-1)-7=0$

3. (a) Solve the equation $y^2-7y+10=0$

(b) Hence find the solutions to $(x^2+1)^2-7(x^2+1)+10=0$

$Ans((a)2,5\ (b)\pm 1,\pm 2)$

4. (a) Solve $y^2-5y-14=0$.

(b) Hence find the solutions to $(x^3-1)^2-5(x^3-1)-14=0$

$Ans((a)(-2,7),(b)(-1,2))$

5. For each of the following find the minimum value of y and state the value of x at which it occurs.

(a) $y=x^2+4x+6$ (b) $y=2x^2+10x-5$ (c) $y=5x^2-2x+8$

$Ans\left((a)y=2,x=-2,(b)y=-17.5,x=-2.5\ (c)y=-7\frac{4}{5},x=\frac{1}{5}\right)$

6. For each of the following find the maximum value of y and state the value of x at which it occurs.

(a) $y=3-2x-x^2$ (b) $y=3+2x-2x^2$ (c) $y=12+7x-14x^2$

$Ans((a)y=6.25,x=-1.5\ (b)y=3.5,x=0.5\ (c)y=12\frac{7}{8},x=0.25)$

7. Use the method of completing squares to sketch the graphs of these quadratics.

(a) $y=x^2-6x+8$ (b) $y=2x^2-4x+5$ (c) $y=3x^2-18+24$

(d) $y=-x^2+2x-3$ (e) $y=15-4x-4x^2$ (f) $y=10+3x-x^2$

8. A farmer has 40m of fencing with which to enclose a rectangular pen. Given the pen is $x\ m$ wide,

(a) Show that its area is $(20x-x^2)m^2$

(b) Deduce the maximum area that he can enclose. $Ans(100\ m^2)$

9. (i) Write $x^2+6x+16$ in the form $(x+a)^2+b$, where a and b are integers to be found.

(ii) Find the minimum value of $x^2+6x+16$ and state the value of x for which this minimum occurs.

(iii) Write down the maximum value of the function $\frac{1}{x^2+6x+16}$

$Ans((x+3)^2+7,(ii)7\ at\ x=-3\ (iii)\ 1/7)$

10. Given that, for all values of x, $3x^2+12x+5\equiv p(x+q)^2+r$

(a) Find the values of p,q and r

(b) Hence, or otherwise, find the minimum value of $3x^2+12x+5$.

(c) Solve the equation $3x^2+12x+5=0$, giving your answer to one decimal place.

$Ans((a)p=3,q=2,r=-7,(b)-7\ (c)-3.5,-0.5)$

Simultaneous equations, one linear and the other nonlinear

Example 47

Solve the simultaneous equations $y=x^2+3x+2$ and $y=2x+8$

Solution

Eliminating y, obtaining

$x^2 + 3x + 2 = 2x + 8$

$\Rightarrow x^2 + x - 6 = 0$

$\Rightarrow (x + 3)(x - 2) = 0$

$\therefore x = -3 \; or \; x = 2$

When $x = -3; y = 2(-3) + 8 = 2$

When $x = 2; y = 2(2) + 8 = 12$

The solutions are $x = -3, y = 2$ and $x = 2, y = 12$

Example 48

Solve the simultaneous equations

$x^2 + 2x - y = 14$

$2x^2 - 3y = 47$

Solution

$x^2 + 2x - y = 14$ ……………..(1)

$2x^2 - 3y = 47$ …………………..(2)

From equation (1) $y = x^2 + 2x - 14$……………(3)

Putting equation (3) into (2)

$\Rightarrow 2x^2 - 3(x^2 + 2x - 14) = 47$

$\Rightarrow 2x^2 - 3x^2 - 6x + 42 - 47 = 0$

$\Rightarrow -x^2 - 6x - 5 = 0$

$\Rightarrow x^2 + 6x + 5 = 0$

$\Rightarrow (x + 5)(x + 1) = 0$

$\therefore x = -5 \; or \; x = -1$

When $x = -5; y = (-5)^2 + 2(-5) - 14 = 1$

When $x = -1; y = (-1)^2 + 2(-1) - 14 = -15$

The solutions are $x = -5, y = 1$and $x = -1, y = -15$.

Example 49

Solve the equations

$x + \frac{1}{y} = 1$

$y + \frac{1}{x} = 4$

Solution

The equations become;

$xy + 1 = y$

$xy + 1 = 4x$

$\Rightarrow y = 4x$and on substitution

$\Rightarrow 4x^2 + 1 = 4x$
$\Rightarrow 4x^2 - 4x + 1 = 0$
$\Rightarrow (2x - 1)^2 = 0 \therefore x = 1/2$
$\Rightarrow y = 4x = 4(1/2) = 2$
$\therefore$the solution is $x = 1/2\,, y = 2$

Exercise 2.2(c)

1. Solve the simultaneous equations;
(a) $x + 2y = 3, x^2 - xy + 5y^2 + 2y = 7$
(b) $x^2 + 2xy = 3, 3x^2 - y^2 = 26$
(c) $x^2 + y^2 = 5, \frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{4}$
2. Solve the simultaneous equations $2x^2 - xy + y^2 = 32$ and $y = -5/x$
$Ans(x = \pm 5/\sqrt{2}\,, y = -\sqrt{2}, x = \pm 1, y = \mp 5)$
3. Solve each of the following pairs of simultaneous equations;
(a) $y = 3x - 4, y = x^2 - 4x + 6$ (b) $xy = 2, 3y - 2x = 11$
$Ans((a)(2,2), (b)(-6, -1/3), (1/2\,, 4))$
4. Solve the simultaneous equations $\frac{x+2}{y-4} + \frac{2(y-4)}{x+2} + 3 = 0, x - y = 3$

The sum and product of the roots of a quadratic equation

- The general quadratic equation $ax^2 + bx + c = 0$ can be written as

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ ……………………….(1)

If α and β are the roots of the equation $ax^2 + bx + c = 0$,

$\Rightarrow (x - \alpha)(x - \beta) = 0$

$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$ …………………….(2)

Since equations (1) and (2) are identical

$\Rightarrow \alpha + \beta = -\frac{b}{a}$…..(sum of roots)……..*

$\Rightarrow \alpha\beta = \frac{c}{a}$ ……..(product of roots)……**

- We can use our knowledge of $(\alpha + \beta)$ and $\alpha\beta$ to find the value of other symmetrical expressions in α and β.
- Also we can find the equation whose roots are known using the formula
$x^2 - (sum\ of\ the\ roots)x + (product\ of\ roots) = 0$
- The following are the useful identities:

(a) From $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$
$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(b) From $(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$\Longrightarrow \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

(c) From $(\alpha-\beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$

$\Longrightarrow (\alpha-\beta) = \sqrt{\alpha^2+\beta^2-2\alpha\beta}$

$= \sqrt{((\alpha+\beta)^2 - 2\alpha\beta) - 2\alpha\beta}$

$= \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

Example 50

Write down the sum and product of the roots of the equations;

(a) $3x^2 - 6x + 2 = 0$
(b) $5x^2 + 11x + 3 = 0$
(c) $x^2 + 5x = 1$

Solution

(a) $3x^2 - 6x + 2 = 0$

The sum of the roots $= -\frac{(-6)}{3} = 2$

The product of the roots $= \frac{2}{3}$

(b) $5x^2 + 11x + 3 = 0$

The sum of the roots $= -\frac{(11)}{5}$

The product of the roots $= \frac{3}{5}$

(c) $x^2 + 5x - 1 = 0$

The sum of the roots $= -5$

The product of the roots $= -1$

Example 51

Write down the quadratic equation, the sum and product of whose roots are $3/4$ and -7 respectively.

Solution

The equation is $x^2 - \frac{3}{4}x + (-7) = 0$ i.e. $4x^2 - 3x - 28 = 0$

Example 52

Given that one root of the equation $3x^2 + 4x + k = 0$ is three times the other, find k.

Solution

Let the roots of the equation be $\alpha$and 3α.

The sum of the roots $= \alpha + 3\alpha = -\frac{4}{3}$ i.e.$4\alpha = -\frac{4}{3} \therefore \alpha = -\frac{1}{3}$

The product of the roots $= \alpha.3\alpha = \frac{k}{3}$
$\Rightarrow 3\alpha^2 = \frac{k}{3}$
$\Rightarrow 3\left(-\frac{1}{3}\right)^2 = \frac{k}{3}; \therefore k = 1$

Example 53

If α and β are the roots of the equation $3x^2 - 5x + 1 = 0$, find the values of;

(a) $\frac{1}{\alpha} + \frac{1}{\beta}$ (b) $\alpha^2 + \beta^2$ (c) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$.

Solution

$\alpha + \beta = \frac{5}{3}$ and $\alpha\beta = \frac{1}{3}$

(a) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta} = \left(\frac{5}{3}\right) \div \left(\frac{1}{3}\right) = 5$
(b) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= \left(\frac{5}{3}\right)^2 - 2\left(\frac{1}{3}\right)$
$= \frac{25}{9} - \frac{2}{3}$
$= \frac{2}{3}$
(c) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{\beta+\alpha}{\alpha^2\beta^2}$
$= \frac{5}{3} \div \left(\frac{1}{3}\right)^2$
$= 15$

Example 54

If α and β are the roots of the equation $x^2 + 7x + 5 = 0$, find an equation whose roots are $\alpha + 1$ and $\beta + 1$

Solution
Method 1:
$\alpha + \beta = -7, \alpha\beta = 5$
Sum of new roots, $(\alpha + 1) + (\beta + 1) = \alpha + \beta + 2$
$= -7 + 2 = -5$
Product of new roots, $(\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1$
$= 5 - 7 + 1$
$= -1$

The required equation is $x^2 - (-5)x + (-1) = 0$ i.e. $x^2 + 5x - 1 = 0$

Method 2:

For the equation with roots $x = \alpha + 1\ or\ x = \beta + 1$

$\Rightarrow x - 1 = \alpha\ or\ x - 1 = \beta$

Since the equation with roots α and $\beta$is $x^2 + 7x + 5 = 0$, the required equation must take the form $(x-1)^2 + 7(x-1) + 5 = 0$

$\Rightarrow x^2 - 2x + 1 + 7x - 7 + 5 = 0$

$\therefore x^2 + 5x - 1 = 0$

Exercise 2.2(d)

1. Find the sums and products of the roots of the following equations.
(a) $2x^2 - 5x - 3 = 0$ (b) $4x - \frac{1}{x} = 3$ (c) $6 + x - x^2$
$Ans((a)\ 5/2\ , -3/2\ (b)\ 3/4\ , -1/4\ (c) 1, -6)$
2. Find the equations, the sum and products of whose roots are, respectively;
(a) $3, 2$ (b) $-1/2\ , 3/4$ (c) $a/b\ , 1/ab$
3. Given that one root of the equation $2x^2 - kx + k = 0$, where $k \neq 0$, is twice the other, find k. $Ans(9)$
4. Given that the two roots of the equation $x^2 + (7-p)x - p = 0$differ by 5,find the possible values of $p.Ans(4, 6)$
5. If α and β are the roots of the equation$ax^2 + bx + c = 0$, prove that;
(i) if$\beta = 4\alpha$ then $4b^2 = 25ac$ (ii) if $\beta = \alpha + 1$ then $a^2 = b^2 - 4ac$
6. If α and β are the roots of the equation $x^2 - 3x - 2 = 0$, find the values of;
(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\alpha^2 + \beta^2$ (d) $\alpha^3 + \beta^3$ (e) $(\alpha - \beta)$ (f) $\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2}$ (g) $\alpha^2 - \beta^2$
7. If $p + q = 5$and $p^2 + q^2 = 19$, find the value of pq and write down an equation in x whose roots are p and q. $Ans(3, x^2 - 5x + 3 = 0)$
8. If $a - b = 3$and $a^2 + b^2 = 65$, write down an equaltion in x whose roots are a and b. $Ans(x^2 \pm 11x + 28 = 0)$
9. If α and β are the roots of the equation $x^2 - 4x + 2 = 0$, find equations whose roots are (a) $3\alpha, 3\beta$ (b) $\alpha + 3, \beta + 3$ (c) $\alpha + 3\beta, \beta + 3\alpha$.
$Ans((a)x^2 - 12x + 18, (b)x^2 - 10x + 23 = 0, (c)x^2 - 16x + 56 = 0)$
10. Given that the equations $2x^2 + 3x + k = 0$ and $3x^2 + x - 2k = 0$ have a common root, find the possible values of k. $Ans(0, 1)$
11. In the equation $ax^2 + bx + c = 0$, one root is the square of the other. Without solving the equation, prove that $c(a-b)^3 = a(c-b)^3$.
12. Prove that if the difference between the roots of the equation $ax^2 - bx + c = 0$ is the same as the difference between the roots of $bx^2 - cx + a = 0$ then $b^4 - a^2c^2 = 4ab(bc - a^2)$.
13. If α and β are the roots of the equation $ax^2 + 2bx + c = 0$. Prove that

$\left(\alpha+\frac{1}{\alpha}\right)^2+\left(\beta+\frac{1}{\beta}\right)^2=\frac{4b^2(a^2+c^2)-2ac(a-c)^2}{a^2c^2}$

14. Find in its simplest rational form, the equation whose roots are $\frac{\sqrt{7}}{(\sqrt{7}\pm\sqrt{5})}$
15. Given that the roots of the equation $x^2+ax+(a+2)=0$ differ by 2, find the possible values of the constant a. Hence state the values of the roots of the equation.

1.3 Other equations:

(a) Condition for common roots:

Example 55

Show that, if the equations $x^2+ax+1=0$ and $x^2+x+b=0$ have a common root, then $(b-1)^2=(a-1)(1-ab)$

Solution

Let the common root be x_1

$x_1^2+ax_1+1=0$ ……………………..(1)

$x_1^2+x_1+b=0$ ………………………(2)

Equation (1)-(2) to eliminate x_1^2

$\Rightarrow (a-1)x_1+1-b=0$

$\Rightarrow x_1=\frac{(b-1)}{(a-1)}$ …………………………(3)

Eliminating x_1; $a\times(2)-(1)$

$\Rightarrow x_1^2(a-1)+(ab-1)=0$

$\Rightarrow x_1^2=\frac{(1-ab)}{(a-1)}$ ……………………… (4)

From (3) and (4)

$\Rightarrow \frac{(b-1)^2}{(a-1)^2}=\frac{1-ab}{a-1}$

$\therefore (b-1)^2=(a-1)(1-ab$ ……………………..■

Exercise 2.3a

1. Show that, if the equations $x^2+2px+q=0, x^2+2Px+Q=0$ have a common root, then, $(q-Q)^2+4(P-p)(Pq-pQ)=0$.
2. Find the condition that the equations $x^2+2x+a=0,\ x^2+bx+3=0$ should have a common root. $Ans((3-a)^2=(6-ab)(b-2))$
3. Prove that, if the equations $x^2+ax+b=0$ and $cx^2+2ax-3b=0$ have a common root and neither a nor b is zero then $b=\frac{5a^2(c-2)}{(c+3)^2}$
4. Show that if the expressions $x^2+bx+c=0$ and $x^2+px+q=0$ have a common factor then $(c-q)^2=(b-p)(cp-bq)$.
5. Given that the equation $y^2+py+q=0$ and $y^2+my+k=0$ have a common root show that $(q-k)^2=(m-p)(pk-mq)$

(b) Radical equations

- Here we use the identity $(a \pm b)^2 = a^2 \pm 2ab + b^2$
- Finally we **must** check to see the most correct answer.

Example 56

Solve the equation $\sqrt{(5x-25)} - \sqrt{(x-1)} = 2$

Solution

$\Rightarrow \sqrt{(5x-25)} = 2 + \sqrt{(x-1)}$

Squaring both sides

$\Rightarrow \left(\sqrt{(5x-25)}\right)^2 = \left(2 + \sqrt{(x-1)}\right)^2$

$\Rightarrow 5x - 25 = 4 + 4\sqrt{(x-1)} + x - 1$

$\Rightarrow 4x - 28 = 4\sqrt{(x-1)}$

$\Rightarrow x - 7 = \sqrt{(x-1)}$

Squaring both sides again

$\Rightarrow (x-7)^2 = \left(\sqrt{(x-1)}\right)^2$

$\Rightarrow x^2 - 14x + 49 = x - 1$

$\Rightarrow x^2 - 15x + 50$

$\Rightarrow (x-5)(x-10) = 0$

$\therefore x = 5 \text{ or } x = 10$

Checking

When $x = 5$

$LHS = \sqrt{(5(5)-25)} - \sqrt{(5-1)} = -2$ Not a solution

When $x = 10$

$LHS = \sqrt{(5(10)-25)} - \sqrt{(10-1)} = 2 = RHS$

$\therefore x = 10$ is the only root of the equation.

Example 57

Solve the equation $\sqrt{(4-x)} - \sqrt{(6+x)} = \sqrt{(14+2x)}$

Solution

Squaring both sides;

$\Rightarrow \left(\sqrt{(4-x)} - \sqrt{(6+x)}\right)^2 = \left(\sqrt{(14+2x)}\right)^2$

$\Rightarrow 4 - x - 2\sqrt{[(4-x)(6+x)]} + 6 + x = 14 + 2x$

$\Rightarrow -2\sqrt{[(4-x)(6+x)]} = 4 + 2x$

$\Rightarrow -\sqrt{[(4-x)(6+x)]} = 2 + x$

Squaring both sides again

$\Rightarrow \left(-\sqrt{[(4-x)(6+x)]}\right)^2 = (2+x)^2$

$\Rightarrow (4-x)(6+x) = 4+4x+x^2$

$\Rightarrow 24-2x-x^2 = 4+4x+x^2$

$\Rightarrow 2x^2+6x-20 = 0$

$\Rightarrow 2(x+5)(x-2) = 0$

$\Rightarrow x = 2 \text{ or } x = -5$

Verifying

When $x = 2$

$LHS = \sqrt{(4-2)} - \sqrt{(6+2)} = \sqrt{2} - \sqrt{8} = -\sqrt{2}$

$RHS = \sqrt{(14+2(2))} = \sqrt{18} = 3\sqrt{2} \neq LHS$

When $x = -5$

$LHS = \sqrt{(4+5)} - \sqrt{(6-5)} = \sqrt{9} - \sqrt{1} = 2$

$RHS = \sqrt{(14+2(-5))} = \sqrt{4} = 2 = LHS$

$\therefore x = -5$ is the only solution.

Exercise 2.3b

1. Solve the following equations;

(a) $\sqrt{(x+1)} + \sqrt{(x-2)} = 3$ $Ans(3)$

(b) $\sqrt{(3x-3)} - \sqrt{x} = 1$ $Ans(4 \text{ not } 1)$

(c) $\sqrt{(x-5)} + \sqrt{x} = 5$ $Ans(9)$

(d) $2\sqrt{(x-1)} + \sqrt{(x-4)} = x$ $Ans(5)$

(e) $2\sqrt{(2x-12)} + \sqrt{(2x-3)} = 3$ $Ans(14 \text{ not } 6)$

(f) $\sqrt{(x-1)} + 2\sqrt{(x-4)} = 4$ $Ans(5 \text{ not } 22\frac{7}{9})$

2. Find the only solution of the equation $\sqrt{(4x-2)} + \sqrt{(x+1)} - \sqrt{(7-5x)} = 0$
   $Ans(1 - \frac{1}{7}\sqrt{7})$
3. Solve $\sqrt{(4x+13)} - \sqrt{(x+1)} = \sqrt{(12-x)}$ $Ans(3)$
4. Solve the following equations;

(a) $2\sqrt{(x+5)} - \sqrt{(2x+8)} = 2$

(b) $\sqrt{(x+1)} + \sqrt{(5x+1)} = 2\sqrt{(x+6)}$

(c) The a quartic equations

- These are equations of the form $Ax^4 + Bx^3 + Cx^2 + Bx + A = 0$;
- The coefficients are arranged symmetrically
- We divide through by x^2 and use a specific substitution.

Example 58

Using the substitution $y = x + \frac{1}{x}$solve the equation $2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0$

Solution

Using $y = x + \frac{1}{x}$;

$\Longrightarrow y^2 = \left(x + \frac{1}{x}\right)^2$

$\Longrightarrow y^2 = x^2 + 2 + \frac{1}{x^2}$; $y - 2 = x^2 + \frac{1}{x^2}$

Dividing through by x^2

$\Longrightarrow 2x^2 - 9x + 14 - \frac{9}{x} + \frac{2}{x^2} = 0$

$\Longrightarrow 2\left(x^2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) + 14 = 0$

$\Longrightarrow 2y^2 - 9y + 14 = 0$

$\Longrightarrow (y - 2)(2y - 5) = 0$

$\Longrightarrow y = 2 \ or \ y = \frac{5}{2}$

When $y = 2$

$\Longrightarrow x + \frac{1}{x} = 2$

$\Longrightarrow x^2 - 2x + 1 = 0$

$\Longrightarrow (x - 1)^2 = 0 \therefore x = 1, 1$

When $y = \frac{5}{2}$

$\Longrightarrow x + \frac{1}{x} = \frac{5}{2}$

$\Longrightarrow 2x^2 - 5x + 2 = 0$

$\Longrightarrow (2x - 1)(x - 2) = 0 \therefore x = 1/2, \ 2$

Therefore the roots of the equation are $1/2, 1, 1, 2$

Exercise 2.3c

1. Solve the following equations;

(a) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0 \ Ans(1/3, 1/2, 2, 3$

(b) $4x^4 + 17x^3 + 8x^2 + 17x + 4 = 0 \ \ Ans(-1/4, -4)$

(c) $x^4 - 4x^3 + 6x^2 - 4x + 1 = 0 \ \ Ans\ (1)$

(d) $x^4 - 2x^3 - 6x^2 - 2x + 1 = 0 \ \ Ans(-1, 2 \pm \sqrt{3})$

(e) $y^4 - 2y^3 - 2y^2 + 2y + 1 = 0 \ \ Ans(\pm 1, 1 \pm \sqrt{2})$

2. Using the substitution $y = x^2 - 4x$, or otherwise find the real roots of the equation $2x^4 - 16x^3 + 77x^2 - 180x + 63 = 0$
3. Solve the equation $2x^4 + x^3 - 6x^2 + x + 2 = 0$
4. If $p = x + \frac{1}{x}$, express $x^2 + \frac{1}{x^2}$, $x^3 + \frac{1}{x^3}$, $x^4 + \frac{1}{x^4}$ in terms of p
5. By putting $z = x + x^{-1}$, solve the equation;
   $2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0 \ \ Ans(1/2, 1, 1, 2)$

(d) Relevant equations

Example 59

Solve the simultaneous equations $x + y + z = 2$, $\frac{x+2y}{-3} = \frac{y+2z}{4} = \frac{2x+z}{5}$

Solution

$x + y + z = 2$..................(1)

From $\frac{x+2y}{-3} = \frac{y+2z}{4} = \frac{2x+z}{5} = k$

$\Rightarrow \frac{3x+3y+3z}{6} = k$

$\Rightarrow \frac{3(x+y+z)}{6} = k$

$\Rightarrow \frac{3(2)}{6} = k;\ \ k = 1$

$\Rightarrow \frac{x+2y}{-3} = 1; x + 2y = -3 \ldots\ldots\ldots\ldots\ldots..(2)$

$\Rightarrow \frac{y+2z}{4} = 1;\ \ y + 2z = 4 \ldots\ldots\ldots\ldots\ldots.(3)$

$\Rightarrow \frac{2x+z}{5} = 1;\ \ 2x + z = 5 \ldots\ldots\ldots\ldots(4)$

From (3) $y = 4 - 2z \ldots\ldots\ldots\ldots..(5)$

Putting (5) into (1) gives

$x + 2(4 - 2z) = -3$

$\Rightarrow x - 4z = -11$ ....(6)

Solving (4) and (6)

$x - 4z = -11$ ....(6)

$2x + z = 5$ ....(4)

$2(6) - (4)$ gives

$-9z = -27;\ \ z = 3$

From (6) $x = -11 + 4(3) = 1$

From (5) $y = 4 - 2(3) = -2$

$\therefore x = 1, y = -2, z = 3$

Exercise 2.3d

1. Solve the following simultaneous equations
   $\frac{x}{1} = \frac{x+y}{3} = \frac{x-y+z}{2}$, $x^2 + y^2 + z^2 + x + 2y + 4z - 6 = 0$ $Ans(2/7, 4/7, 6/7)$
2. Solve the simultaneous equations $\frac{x-y}{4} = \frac{z-y}{3} = \frac{2z-x}{1}$, $x + 3y + 2z = 4$

(e) The square root of $(\boldsymbol{a} + \sqrt{\boldsymbol{b}})$

- Here w engage the skills of solving simultaneous equations that we have been using before.
- Note the following;

(i) $\sqrt{(a + \sqrt{b})} = \pm(\sqrt{x} + \sqrt{y})$ (ii) $\sqrt{(a - \sqrt{b})} = \pm(\sqrt{x} - \sqrt{y})$

Example 60

Find the square root of $14 + 6\sqrt{5}$

Solution

Let $\sqrt{(14 + 6\sqrt{5})} = \pm(\sqrt{x} + \sqrt{y})$

Squaring both sides;

$\Rightarrow 14 + 6\sqrt{5} = x + 2\sqrt{xy} + y$

Equating rational and irrational terms;

$\Rightarrow x + y = 14;\ y = 14 - x$……………….(1)

$\Rightarrow 2\sqrt{xy} = 6\sqrt{5}\ ; xy = 45$…………….(2)

Putting equation (1) into (2)

$\Rightarrow x(14 - x) = 45$

$\Rightarrow x^2 - 14x + 45 = 0$

$\Rightarrow (x - 9)(x - 5) = 0$

$\therefore x = 9\ or\ 5\ and\ y = 5\ or\ 9$

$\therefore \sqrt{(14 + 6\sqrt{5})} = \pm(\sqrt{9} + \sqrt{5}) = \pm(3 + \sqrt{5})$

Exercise 2.3e

1. Find the square root of $5 + 2\sqrt{6}$ $Ans(\pm(\sqrt{3} + \sqrt{2}))$
2. Express the square root of $18 - 12\sqrt{2}$ in the form $\sqrt{x} - \sqrt{y}$ where x and y are rational numbers. $Ans(\pm(\sqrt{12} - \sqrt{6}))$
3. Find rational numbers a and b such that $3 + \sqrt{2} = (a + b)(6 - \sqrt{2})^2$
   $Ans(69/578\,, 37/578)$

CHAPTER 3
POLYNOMIALS
3.1 Operation on polynomials

- An expression of the form $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$ where $a_n, a_{n-1}, \ldots, a_0$ are real numbers with $a_n \neq 0$ and n is a positive integer, is called a **polynomial of degree n.**
- When $n = 2$, the polynomial is called a **quadratic.**
- When $n = 3$, the polynomial is called a **cubic.**
- When $n = 4$, the polynomial is called a **quartic.**

Example 61

Find the degree of each of these polynomials

(a) $4x^6 + 3x^5 + x^3 - x^2 + 5x$ (b) $x^4 - 3x^3 + 2x^9 - 7$

Solution

(a) The highest power of x that occurs is 6. Therefore, the degree of the polynomial is 6.
(b) Rearranging the terms in descending order gives
$2x^9 + x^4 - 3x^3 - 7$
The highest power of x that occurs is 9. Therefore, the degree of the polynomial is 9.

Example 62

Given the polynomial $f(x) = 3x^4 + 2x^2 - x + 7$, evaluate;

(a) $f(0)$ (b) $f(-2)$

Solution

(a) $f(0) = 3(0)^4 + 2(0)^2 - (0) + 7 = 7$
(b) $f(-2) = 3(-2)^4 + 2(-2)^2 - (-2) + 7 = 65$

Addition and subtraction of polynomials

- Polynomials can be added or subtracted by collecting together terms of the same degree. In general the result of adding or subtracting is also a polynomial.

Example 63

Given the two polynomials $f(x) = 3x^3 + 2x^2 - x + 4$and $g(x) = x^3 - x^2 + 7$.

Find;
(a) $f(x) + g(x)$ (b) $f(x) - g(x)$

Solution

(a) $f(x) + g(x) = (3x^3 + 2x^2 - x + 4) + (x^3 - x^2 + 7)$
$= (3x^3 + x^3) + (2x^2 - x^2) + (-x) + (4 + 7)$

$= 4x^3 + x^2 - x + 11$

(b) $f(x) - g(x) = (3x^3 + 2x^2 - x + 4) - (x^3 - x^2 + 7)$

$= (3x^3 - x^3) + (2x^2 + x^2) + (-x) + (4 - 7)$

$= 2x^3 + 3x^2 - x - 3$

Multiplication of polynomials

- When two polynomials are multiplied together each term of one polynomial is multiplied by each term of the other polynomial. In general, the product of any number of polynomial is also a polynomial.

Example 64

Given the two polynomials $f(x) = x^4 + 4x - 1$and $g(x) = 2x^4 + x^3 - 4x$, find in simplest form each of the polynomials.

(a) $f(x)g(x)$ (b) $3f(x) + 4xg(x)$

Solution

(a) $f(x)g(x) = (x^4 + 4x - 1)(2x^4 + x^3 - 4x)$

$= x^4(2x^4 + x^3 - 4x) + 4x(2x^4 + x^3 - 4x) - 1(2x^4 + x^3 - 4x)$

$= 2x^8 + x^7 - 4x^5 + 8x^5 + 4x^4 - 16x^2 - 2x^4 - x^3 + 4$

$= 2x^8 + x^7 + 4x^5 + 2x^4 - x^3 - 16x^2 + 4x$

(b) $3f(x) + 4xg(x) = 3(x^4 + 4x - 1) + 4x(2x^4 + x^3 - 4x)$

$= 3x^4 + 12x - 3 + 8x^5 + 4x^4 - 16x^2$

$= 8x^5 + 7x^4 - 16x^2 + 12x - 3$

Dividing polynomials

- Before we look at the technique of dividing polynomials, it will be useful to recall a technique used in the division of numbers.
- One way of dividing '19 divided by 5' is
  $\frac{19}{5} = 3\ remainder\ 4 \quad or 19 = 5 \times \left(\begin{matrix}3\\ called\ the\ quotient\end{matrix}\right) + \left(\begin{matrix}4\\ called\ the\ remainder\end{matrix}\right)$
- The same method is also applied to polynomial division.

Example 65

Find the quotient and the remainder when the polynomial $x^2 + 4x - 5$ is divided by $x + 3$

Solution

Let the quotient be $(ax + b)$ and the remainder be r

$\Longrightarrow x^2 + 4x - 5 \equiv (x + 3)(ax + b) + r$

Expanding and collecting like terms

$\Longrightarrow x^2 + 4x - 5 \equiv ax^2 + bx + 3ax + 3b + r$

$\equiv ax^2 + (b + 3a)x + (3b + r)$

Comparing the coefficients of the x^2 term gives

$a = 1$

Comparing the coefficients of the x term gives

$3a + b = 4; \quad 3 + b = 4; \quad \therefore b = 1$

Comparing the constant terms gives

$-5 = 3b + r$; $-5 = 3(1) + r$; $\therefore r = -8$
Therefore the quotient is $x + 1$ and the remainder is -8.
Note: the symbol $\equiv$, which means 'is identically equal to'. Strictly speaking $\equiv$ should always be used instead of $=$ when dealing with identities.

Example 66

(a) Find the remainder when the polynomial $x^3 + x^2 - 14x - 24$ is divided by

(i) $(x + 1)$ (ii) $x + 3$
(b) Hence factorize $x^3 + x^2 - 14x - 24$
Solution
(a) (i) let the quotient and the remainder be $(ax^2 + bx + c)$ and r respectively
$\Rightarrow x^3 + x^2 - 14x - 24 \equiv (x + 1)(ax^2 + bx + c) + r$
$\Rightarrow x^3 + x^2 - 14x - 24 \equiv ax^3 + bx^2 + cx + ax^2 + bx + c + r$
$\equiv ax^3 + (a + b)x^2 + (b + c)x + (c + r)$
Comparing the coefficients of;
x^3 ; $a = 1$
x^2 ; $a + b = 1$; $1 + b = 1$; $\therefore b = 0$
x; $b + c = -14$; $\therefore c = -14$
x^0; $c + r = -24$; $-14 + r = -24$; $\therefore r = -10$
When $x^3 + x^2 - 14x - 24$ is divded by $x + 1$ the remainder is -10

(ii) $\Rightarrow x^3 + x^2 - 14x - 24 \equiv (x + 3)(ax^2 + bx + c) + r$
$\equiv ax^3 + (3a + b)x^2 + (3b + c)x + (3c + r)$

Comparing the coefficients of;
x^3 ; $a = 1$
x^2 ; $3a + b = 1$; $3 + b = 1$; $\therefore b = -2$
x; $3b + c = -14$; $3(-2) + c = -14$ $\therefore c = -8$
x^0; $3c + r = -24$; $3(-8) + r = -24$; $\therefore r = 0$
When $x^3 + x^2 - 14x - 24$ is divded by $x + 3$ the remainder is 0
(b) Since the remainder is 0 when $x^3 + x^2 - 14x - 24$ is divided by $(x + 3)$, then $(x + 3)$ is a factor of $x^3 + x^2 - 14x - 24$
We need to factorize $x^2 - 2x - 8$
$\Rightarrow x^2 - 2x - 8 = (x + 2)(x - 4)$
We have $x^3 + x^2 - 14x - 2 = (x + 3)(x + 2)(x - 4)$

3.2 The remainder theorem

- When the polynomial $f(x)$ is divided by $(\alpha x - \beta)$, the remainder is $f\left(\frac{\beta}{\alpha}\right)$

Example 67

Find each of the remainders when the polynomial $x^3 + 5x^2 - 17x - 21$ is divided by (a) $x + 1$ (b) $2x + 1$

Solution

(a) Let $f(x) = x^3 + 5x^2 - 17x - 21$; let $x + 1 = 0; x = -1$

$\Rightarrow f(-1) = (-1)^3 + 5(-1)^2 - 17(-1) - 21$

$= -1 + 5 + 1 - 21$

$= 0$

Therefore the remainder is 0

(b) Let $2x + 1 = 0; x = -1/2$

$\Rightarrow f(-1/2) = (-1/2)^3 + 5(-1/2)^2 - 17(-1/2) - 21$

$= -\frac{1}{8} + \frac{5}{4} + \frac{17}{2} - 21$

$= -\frac{91}{8}$

Therefore the remainder is $-\frac{91}{8}$

Example 68

Find the remainder when the polynomial $f(x) = 2x^3 + 5x^2 - 39x + 18$ is divided by $(x + 6)$. Hence solve the equation $2x^3 + 5x^2 - 39x + 18 = 0$.

Solution

$\Rightarrow f(-6) = 2(-6)^3 + 5(-6)^2 - 39(-6) + 18$

$= -432 + 180 + 234 + 18$

$= 0$

Since the remainder is 0, $(x + 6)$ is a factor of $f(x)$ and

$2x^3 + 5x^2 - 39x + 18 \equiv (x + 6)(ax^2 + bx + c)$

Expanding and comparing coefficients

$a = 2,\ b = -7$and $c = 3$. Therefore

$2x^3 + 5x^2 - 39x + 18 \equiv (x + 6)(2x^2 - 7x + 3)$

$f(x) = (x + 6)(2x - 1)(x - 3)$

$0 = (x + 6)(2x - 1)(x - 3)$

Solving gives $x = -6, x = 1/2 , x = 3$

Example 69

Given that when the polynomial $f(x) = x^3 + ax^2 + bx + 2$ is divided by $x - 1$ the remainder is 4 and when it is divided by $x + 2$ the remainder is also 4, find the values of the constants $a$and b.

Solution

By the remainder theorem, $f(1) = 4$

$\Rightarrow (1)^3 + a(1)^2 + b(1) + 2 = 4$

$\Rightarrow a + b = 1$ ……………………………………….(1)
Also by the remainder theorem, $f(-2) = 4$
$\Rightarrow (-2)^3 + a(-2)^2 + b(-2) + 2 = 4$
$\Rightarrow 4a - 2b = 10$
$\Rightarrow 2a - b = 5$ ……………………………………(2)
Solving equations (1) and (2) simultaneously gives $a = 2$ and $b = -1$

Factorising polynomials
We use factorisation method to solve cubic equations.
The factor theorem:
If $\alpha x - \beta$ is a factor of the polynomial $f(x)$, then $f\left(\frac{\beta}{\alpha}\right) = 0$

Example 70

Factorise the cubic $f(x) = x^3 + 3x^2 - 13x - 15$

Solution
We know that if $f(x)$ has a linear factor $\alpha x - \beta$, the constant β will be a factor of 15.
$f(-1) = 0$, meaning that $x + 1$ is a factor of $f(x)$.
$\Rightarrow x^3 + 3x^2 - 13x - 15 \equiv (x + 1)(ax^2 + bx + c)$
Expanding and comparing coefficients by inspection, we get $a = 1, b = 2$and $c = -15$.
$\Rightarrow x^3 + 3x^2 - 13x - 15 \equiv (x + 1)(x^2 + 2x - 15)$
$\equiv (x + 1)(x + 5)9x - 3)$

Example 71

Find using long division the remainder when the polynomial $x^3 - 3x^2 + 6x + 5$ is divided by $x - 2$.

Solution
$x^2 - x + 4$
$x - 2 \overline{) x^3 - 3x^2 + 6x + 5}$
$x^3 - 2x^2$
$-x^2 + 6x$
$-x^2 + 2x$
$4x + 5$
$4x - 8$
13
∴the remainder is 13

Example 72

When the expression $x^5 + 4x^2 + ax + b$ is divided by $x^2 - 1$, the remainder is $2x + 3$. Find the values of $a$and b.

Solution

$\Rightarrow x^5 + 4x^2 + ax + b = (x^2 - 1)Q(x) + (2x + 3)$; where$Q(x)$ is the quotient.

Putting $x = 1; 1 + 4 + a + b = 2 + 3;\ a + b = 0 \ldots$ ....(1)

Putting $x = -1; -1 + 4 - a + b = -2 + 3; -a + b = -2$..........(2)

Equation (1) + (2)

$\Rightarrow 2b = -2$; $b = -1$and $a = 1$

Example 73

A polynomial $f(x)$ leaves a remainder 19 when divided by $(x - 7)$ and 1 when divided by $(x + 2)$. Show that the polynomial $f(x)$leaves a remainder of $2x + 5$ when divided by $x^2 - 5x - 14$.

Solution

$\Rightarrow \frac{f(x)}{x-7} = Q_1(x) + \frac{19}{x-7}$ ....(1)

$\Rightarrow \frac{f(x)}{x+2} = Q_2(x) + \frac{1}{x+2}$ ....(2)

(1)-(2) gives

$\frac{f(x)}{x-7} - \frac{f(x)}{x+2} = Q(x) + \frac{19}{x-7} - \frac{1}{x+2}$

$\Rightarrow \frac{f(x)[x+2-(x-7)]}{x^2-5x-14} = Q(x) + \frac{19x+38-x+7}{x^2-5x-14}$

$\Rightarrow \frac{f(x)[9]}{x^2-5x-14} = Q(x) + \frac{18x+45}{x^2-5x-14}$

$\Rightarrow \frac{f(x)}{x^2-5x-14} = \frac{Q(x)}{9} + \frac{2x+5}{x^2-5x-14}$

$\therefore R(x) = 2x + 5$ is the required remainder

Example 74

When a polynomial $p(x)$ is divided by $x^2 - 5x - 14$, the remainder is given by $2x + 5$. Find the remainder when $p(x)$ is divided by

(a) $x - 7$ (b) $x + 2$

Solution

Let a and b be the remainders when $p(x)$ is divided by $(x - 7)$ and $(x + 2)$ respectively.

$\Rightarrow \frac{p(x)}{x-7} = Q_1(x) + \frac{a}{x-7}$ ....(1)

$\Rightarrow \frac{p(x)}{x+2} = Q_2(x) + \frac{b}{x+2}$ ....(2)

(1)-(2) gives

$\frac{p(x)}{x-7} - \frac{p(x)}{x+2} = Q(x) + \frac{a}{x-7} - \frac{b}{x+2}$

$\Longrightarrow \frac{p(x)[x+2-(x-7)]}{x^2-5x-14} = Q(x) + \frac{ax+2a-bx+7b}{x^2-5x-14}$

$\Longrightarrow \frac{p(x)[9]}{x^2-5x-14} = Q(x) + \frac{(a-b)x+(2a-7b)}{x^2-5x-14}$

$\Longrightarrow \frac{p(x)}{x^2-5x-14} = \frac{Q(x)}{9} + \frac{\frac{(a-b)x+(2a-7b)}{9}}{x^2-5x-14}$

Comparing the coefficients of the given remainder $(2x + 5)$

$\Longrightarrow \frac{a-b}{9} = 2;\ \ a - b = 18 \ldots\ldots\ldots\ldots\ldots\ldots..(3)$

$\Longrightarrow \frac{2a-7b}{9} = 5;\ \ 2a - 7b = 45 \ldots\ldots\ldots\ldots\ldots\ldots..(4)$

Solving (3) and (4) simultaneously

$2(3) - (4)$

$\Longrightarrow b = 1$ and $a = 19$

Exercise 3.1

1. Find the degree of each of the following polynomials;
(a) $2x^3 + 3x^2 - 2x + 4$ (b) $4x - 3x^4$ (c) $5x + 7$
2. Given;
(a) $f(x) = x^2 + 3x + 4$, evaluate $f(2)$
(b) $g(x) = 5x^5 + 2x - 1$, evaluate $f(-1)$
3. Given $f(x) = x^3 + 2x^2 - 3x + 2$ and $g(x) = 2x^3 - x^2 + 5x - 4$ find;
(a) $f(x) + g(x)$ (b) $f(x) - 3g(x)$ (c) $f(x)g(x)$ (d) $f(x) - xg(x)$
4. Find the quotient and the remainder when
(a) $x^2 + 6x + 5$ is divided by $(x + 2)$
(b) $6x^2 - x + 2$ is divided by $(2x + 1)$
(c) $x^3 + 3x^2 - 4x + 1$ is divided by $(x - 2)$
5. The expression $2x^3 - 3x^2 + ax - 5$ gives a remainder of 7 when divided by $x - 2$. Find the value of the constant a. $Ans(a = 4)$
6. The remainder when $x^3 - 2x^2 + ax + 5$ is divided by $x - 3$ is twice the remainder when the same expression is divided by $x + 1$. Find the value of the constant a. $Ans(a = -2)$
7. Given that $x - 2$ is a factor of $x^3 + 4x^2 - 2x + k$, find the value of$k$ $Ans\ (k = -20)$.
8. The expression $2x^3 + 3x^2 + ax + b$ leaves a remainder of 7 when divided by $x - 2$, and a remainder of -3 when divided by $x - 1$. Find the values of the constants a and b. $Ans(a = -13, b = 5)$
9. Given that $x^2 - 4$ is a factor of the cubic $x^3 + cx^2 + dx - 12$, find the values of the constants c and d, and hence factorise the cubic. $Ans(c = 3, d = -4; (x - 2)(x + 2)(x + 3))$

10. The remainder when the expression $x^3 - 2x^2 + ax + b$ is divided by $x - 2$ is five times the remainder when the same expression is divided by $x - 1$, and 12 less than the remainder when the same expression is divided by $x - 3$. Find the values of the constants $a$and b. $Ans(a = 3, b = -1)$
11. Factorise each of these expression;
(a) $x^3 - x^2 - 9x + 9$ (b) $x^4 + 2x^3 - 7x^2 - 8x + 12$ (c) $x^3 + 7x^2 - x - 7$.
12. Find the real solutions to each of the following equations
(a) $x^3 + x^2 - 10x + 8 = 0$ (b) $x^3 + 8x^2 = 2 - 11x$ (c) $x^3 + 5 = 8x - 2x^2$
13. Given that the expression $ax^3 + 8x^2 + bx + 6$ is exactly divisible by $x^2 - 2x - 3$, find the values of a and b. $Ans(a = -5, b = 19)$
14. The cubic polynomial $x^3 + Ax - 12$ is exactly divisible by $x + 3$. Find the constant A, and solve the equation $x^3 + Ax - 12 = 0$ for this value of A.
$Ans(A = -13;\ -3, -1, 4)$
15. Show that $12x^3 + 16x^2 - 5x - 3$ is divisible by $(2x - 1)$ and find the factors of the expression.
16. Find the values of a and b if $ax^4 + bx^3 - 8x^2 + 6$ has reminader $2x + 1$ when divided by $x^2 - 1$. $Ans(a = 3, b = 2)$
17. The expression $px^4 + qx^3 + 3x^2 - 2x + 3$ has remainder $x + 1$ when divided by $x^2 - 3x + 2$. Find the values of $p$and q. $Ans(p = 1, q = -3)$.
18. The expression $ax^2 + bx + c$ is divisible by $x - 1$, has remainder 2 when divided by $x + 1$, and has remainder 8 when divided by $x - 2$. Find the values of a, b, c. $Ans(a = 3, b = -1, c = -2)$
19. $x - 1$and$x + 1$ are factors of the expression $x^3 + ax^2 + bx + c$, and it leaves a remainder of 12 when divided by $x - 2$. Find the values of a, b, c.
$Ans(a = 2, b = -1, c = -2)$
20. Show that $3x^3 + x^2 - 8x + 4$ is zero when $x = 2/3$, and hence factorise the expression. $Ans((x - 1)(x + 2)(3x - 2)$
21. What is the value of a if $2x^2 - x - 6, 3x^2 - 8x + 4$ and $ax^3 - 10x - 4$ have a common factor? $Ans(a = 3)$
22. Find the values of p and q which make $x^4 + 6x^3 + 13x^2 + px + q$ a perfect square.$Ans(p = 12, q = 4)$
23. If $4x^3 + kx^2 + px + 2$ is divisible by $x^2 + \lambda^2$, prove that $kp = 8$.
24. A polynomial expression $P(x)$, when divided by $(x - 1)$ leaves a remainder 3 and, when divided by $(x - 2)$ leaves a remainder 1. Show that when divided by $(x - 1)(x - 2)$ it leaves a remainder $-2x + 5$.
25. Given that $x^3 + x - 10 = 0$;
(a) Show that $x = 2$ is a root of the equation.
(b) Deduce the values of $\alpha + \beta$ and $\alpha\beta$ where α and β are the roots of the equation. Hence form the equation whose roots are $\alpha^2$and β^2.
26. Using log division find the remainder when $x^5 + 4x^2 + 2x + 3$ is divided by
(a) $(x - 1)$ (b) $x + 2)$

27. A polynomial, $f(x)$ when divided by $(x-1)$ leaves a remainder 7 and when divided by $(x-2)$, leaves a remainder 39. Find the remainder when it is divided by $(x-1)(x-2)$.
28. Given the polynomial $p(x)=(x-1)(x-b)Q(x)+R(x)$ where $Q(x)$ is the quotient and $R(x)$ is the remainder, show that; $R(x)=\frac{(x-1)p(b)-(x-b)p(a)}{(b-a)}$. Hence find the remainder when $p(x)$ is divided by x^2-25, given that $p(x)$ divided by $(x-5)$ is 1 and when divided by $(x+5)$ is -3.

CHAPTER 4

PARTIAL FRACTIONS

4.1 Algebraic fractions

- Earlier on in algebra we have been simplifying an expression such as $\frac{1}{x-1}-\frac{1}{x+1}$ by reducing it to $\frac{2}{x^2-1}$.
- We have now reached the stage when the reverse process is of value.
- Given a fraction such as $\frac{5}{x^2+x-6}$ whose denominator factorises, we may split it up into its component fractions, writing it as $\frac{1}{x-2}-\frac{1}{x+3}$; it is now said to be in **partial fractions.**

Example 75

Express $\frac{4}{x+6}-\frac{2}{x+7}$ as a single fraction.

Solution

$$\frac{4}{x+6}-\frac{2}{x+7}\equiv\frac{4(x+7)-2(x+6)}{(x+6)(x+7)}$$
$$\equiv\frac{4x+28-2x-12}{(x+6)(x+7)}$$
$$\equiv\frac{2x+16}{(x+6)(x+7)}$$
$$\therefore\frac{4}{x+6}-\frac{2}{x+7}\equiv\frac{2x+16}{(x+6)(x+7)}$$

Example 76

Express $\frac{2x}{x^2+3x+2}+\frac{3}{x+1}$ as a single fraction.

Solution

$\Longrightarrow x^2+3x+2\equiv(x+1)(x+2)$

Lcm= $(x+1)(x+2)$

$$\frac{2x}{x^2+3x+2}+\frac{3}{x+1}\equiv\frac{2x}{(x+1)(x+2)}+\frac{3}{(x+1)}$$
$$\equiv\frac{2x+3(x+2)}{(x+1)(x+2)}$$
$$\equiv\frac{5x+6}{(x+1)(x+2)}$$
$$\therefore\frac{2x}{x^2+3x+2}+\frac{3}{x+1}\equiv\frac{5x+6}{(x+1)(x+2)}$$

Example 77

Express $x+7+\frac{1}{x-4}-\frac{5}{x+1}$ as a single fraction.

Solution

$$x+7+\frac{1}{x-4}-\frac{5}{x+1}\equiv\frac{(x+7)(x-4)(x+1)+(x+1)-5(x-4)}{(x-4)(x+1)}$$
$$\equiv\frac{x^3+4x^2-25x-28+x+1-5x+20}{(x-4)(x+1)}$$
$$\equiv\frac{x^3+4x^2-29x-7}{(x-4)(x+1)}$$
$$\therefore x+7+\frac{1}{x-4}-\frac{5}{x+1}\equiv\frac{x^3+4x^2-29x-7}{(x-4)(x+1)}$$

Example 78

Express $3\frac{3}{5}+\frac{4}{x+7}$ as a single fraction. Hence solve the equation $3\frac{3}{5}+\frac{4}{x+7}=\frac{8}{5-x}$

Solution

$$3\frac{3}{5}+\frac{4}{x+7}\equiv\frac{18}{5}+\frac{4}{x+7}$$
$$\equiv\frac{18(x+7)+4(5)}{5(x+7)}$$
$$\equiv\frac{18x+126+20}{5(x+7)}$$
$$\equiv\frac{18x+146}{5(x+7)}$$

Using this result, the equation

$$3\frac{3}{5}+\frac{4}{x+7}=\frac{8}{5-x}$$

becomes

$$\frac{18x+146}{5(x+7)}=\frac{8}{5-x}$$

Cross-multiplying and simplifying gives

$$(18x+146)(5-x)=40(x+7)$$
$$\Rightarrow 18x^2+96x-450=0$$
$$\Rightarrow 6(3x+25)(x-3)=0$$
$$\therefore x=-\frac{25}{3}\text{and} x=3$$

Exercise 4.1

1. Express the following as single fractions;

(a) $\frac{3}{x-1}+\frac{2}{x+3}$ (b) $\frac{2}{x+4}-\frac{1}{x-3}$ (c) $\frac{5x}{x^2+3}+\frac{6}{2x-5}$ (d) $\frac{2x+1}{x-2}+\frac{3x}{x+4}$ (e) $\frac{2x}{x+2}-\frac{1}{x-2}$

2. Express each of these as a single fraction

(a) $\frac{x}{x^2+3x-4}+\frac{2}{x-1}$ (b) $x+3+\frac{1}{x-2}-\frac{3}{x+4}$ (c) $1+\frac{3}{x^2-3x+2}+\frac{2}{x-1}$

3. Show that $\frac{3}{x-2}+\frac{4}{x-1}\equiv\frac{7x-11}{(x-2)(x-1)}$. Hence solve the equation $\frac{3}{x-2}+\frac{4}{x-1}=\frac{7}{x}$
4. Express $\frac{3}{x-2}+\frac{x}{x+1}$ as a single fraction. Hence solve the equation $\frac{3}{x-2}+\frac{x}{x+1}=2\frac{1}{7}$
5. Given $\frac{a}{6x-1}-\frac{1}{3x+1}\equiv\frac{b}{(6x-1)(3x+1)}$ where a and b are constants, find the values of a and b.$Ans(a=2, b=3)$
6. Given $\frac{6}{x+3}+\frac{P}{x-1}=\frac{Qx}{(x+3)(x-1)}$ where P and Q are both constants, find the values of P and Q.$Ans(P=2, Q=8)$
7. Find the values of the constants a and b for which $\frac{a}{x+b}+\frac{2}{x-4}\equiv\frac{3x}{(x+b)(x-4)}$. $Ans(a=1, b=2)$.

4.2 Partial fractions

- There are basically four different types of algebraic fraction which can be expressed in partial fractions.
- We must check whether the values obtained are true by either
  (i) getting the LCM to go back to the original fraction or
  (ii) substituting any value of x on both sides of the equation.

Type 1: Denominator with linear factors.

- Each linear factor $(ax+b)$ in the denominator has a corresponding partial fraction of the form $\frac{A}{(ax+b)}$ where a, b and A are constants.

Example 79

Express $\frac{7x+8}{(x+4)(x-6)}$ in partial fractions.

Solution

Let $\frac{7x+8}{(x+4)(x-6)}\equiv\frac{A}{(x+4)}+\frac{B}{(x-6)}$

Multiplying through by $(x+4)(x-6)$ gives

$7x+8\equiv A(x-6)+B(x+4)$

Putting $x-6=0, x=6$

$\Rightarrow 7(6)+8=B(6+4); 50=10B,\ B=5$

Putting $x+4=0; x=-4$

$\Rightarrow 7(-4)+8=A(-4-6);\ -20=-10A,\ A=2$

$\therefore \frac{7x+8}{(x+4)(x-6)}=\frac{2}{(x+4)}+\frac{5}{(x-6)}$

Example 80

Express $\frac{9x^2+34x+14}{(x+2)(x^2-x-12)}$ in partial fractions

Solution

$x^2 - x - 12 = (x+3)(x-4)$

Therefore

$\frac{9x^2+34x+14}{(x+2)(x^2-x-12)} \equiv \frac{9x^2+34x+14}{(x+2)(x+3)(x-4)}$

Let $\frac{9x^2+34x+14}{(x+2)(x+3)(x-4)} \equiv \frac{A}{(x+2)} + \frac{B}{(x+3)} + \frac{C}{(x-4)}$

Putting $x = -2\ in\ \frac{9x^2+34x+14}{(x+3)(x-4)}\ gives\ A = \frac{9(-2)^2+34(-2)+14}{((-2)+3)((-2)-4)} = 3$

Putting $x = -3\ in\ \frac{9x^2+34x+14}{(x+2)(x-4)}\ gives\ B = \frac{9(-3)^2+34(-3)+14}{((-3)+3)((-3)-4)} = -1$

Putting $x = 4\ in\ \frac{9x^2+34x+14}{(x+2)(x+3)}\ gives\ C = \frac{9(4)^2+34(4)+14}{((4)+3)((4)-4)} = 7$

$\therefore \frac{9x^2+34x+14}{(x+2)(x^2-x-12)} = \frac{3}{(x+2)} - \frac{1}{(x+3)} + \frac{7}{(x-4)}$

Exercise 4.2a

1. Express the following into partial fractions

(a) $\frac{6}{(x+3)(x-3)} Ans(A = -1, B = 1)$

(b) $\frac{22-16x}{(3+x)(2-x)(4-x)} Ans(A = 2, B = -1, C = 3)$

(c) $\frac{x}{25-x^2} Ans(A = 1/2\,, B = -1/2)$

(d) $\frac{5x+7}{(x+1)(x+2)} Ans(A = 2, B = 3)$

(e) $\frac{3x+2}{(x-1)(x+1)} Ans(A = 5/2\,, B = 1/2)$

(f) $\frac{x^2}{(x-1)(x2)(x-3)} Ans(1/2\,, B = -4, C = 9/2)$

Type II: Denominator with an irreducible quadratic factor.

- Each quadratic factor $(ax^2 + bx + c)$in the denominator which is irreducible, i.e. will not factorise, has a corresponding partial fraction of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where a, b, c, A and B are constants.

Example 81

Express $\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)}$ in partial fractions

Solution

Since $x^2 + 3x + 5$ cannot be factorised, we assume that

$\frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} \equiv \frac{A}{(x-6)} + \frac{Bx+C}{(x^2+3x+5)}$

Multiplying through by $(x-6)(x^2+3x+5)$ gives

$7x^2 + 2x - 28 \equiv A(x^2 + 3x + 5) + (Bx + C)(x - 6)$

Let $x = 6; 7(6)^2 + 2(6) - 28 = A[(6)^2 + 3(6) + 5]$

$236 = 59A;\ \ A = 4$

Comparing coefficients of x^2;

$\Rightarrow 7 = A + B$

$\Rightarrow 7 = 4 + B;\ \ B = 3$

Comparing constant terms;

$\Rightarrow -28 = 5A - 6C$

$\Rightarrow -28 = 5(4) - 6C; \quad C = 8$

$\therefore \frac{7x^2+2x-28}{(x-6)(x^2+3x+5)} \equiv \frac{4}{(x-6)} + \frac{3x+8}{(x^2+3x+5)}$

Exercise 4.2b

1. Express the following into partial fractions

(a) $\frac{3x+1}{(x+1)(x^2+1)} Ans(A = -1, B = 1, C = 2)$

(b) $\frac{3x+4}{(x-2)(x^2+x+1)} Ans(A = 10/7, B = -10/7, C = -9/7)$

(c) $\frac{3x+1}{x(x^2+1)}$

(d) $\frac{x+2}{(x-2)9x^2+4)}$

(e) $\frac{3x-4}{(x+2)(x^2+5x+1)}$

(f) $\frac{x+1}{x^3-1}$

Type III: Denominator with a repeated factor:

- Each repeated linear factor $(ax + b)^2$ in the denominator has corresponding partial fractions of the form $\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2}$ where a, b, A and B are constants.

Example 82

Express $\frac{2x^2+29x-11}{(2x+1)(x-2)^2}$ in partial fractions

Solution

Let $\frac{2x^2+29x-11}{(2x+1)(x-2)^2} \equiv \frac{A}{(2x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2))^2}$

Multiplying through by $(2x+1)(x-2)^2$

$\Rightarrow 2x^2 + 29x - 11 \equiv A(x-2)^2 + B(2x+1)(x-2) + C(2x+1)$

Let $-1/2$; $2(-1/2)^2 + 29(-1/2) - 11 = A(-1/2-2)^2$;

$\Rightarrow -25 = \frac{25}{4}A; A = -4$

Let $x = 2$; $\quad 2(2)^2 + 29(2) - 11 = C(2(2)+1)$

$\Rightarrow 55 = 5C; \quad C = 11$

Comparing the coefficients of the x^2 terms gives;

$2 = A + 2B$

$\Rightarrow 2 = -4 + 2B \ ; \quad B = 3$

$\therefore \frac{2x^2+29x-11}{(2x+1)(x-2)^2} \equiv \frac{-4}{(2x+1)} + \frac{3}{(x-2)} + \frac{11}{(x-2))^2}$

Exercise 4.2c

1. Express the following as partial fractions;

(a) $\frac{1-8x-x^2}{(x+1)(x-1)^2}$ $Ans(A = 2, B = -3, C = -1)$

(b) $\frac{x+1}{(x^2+1)(x-1)^3}$ $Ans(A = 0, B = 1/2\,, C = 0, D = -1/2\,, E = 1)$

(c) $\frac{x+3}{(x+1)(x-1)^2}$

(d) $\frac{1}{(x^2+1)(x-1)^2}$

Type IV: Improper fractions:

- An improper fraction is one in which the degree of the numerator is greater than or equal to the degree of the denominator.
- To simplify an improper algebraic fraction, we divide the numerator (a polynomial) by the denominator (a polynomial).
- When a polynomial of degree n is divided by a polynomial also of degree n, the quotient is a constant.
- When a polynomial of degree n is divided by a polynomial of degree m, where $m > n$, the quotient is a polynomial of degree $n - m$.
- We can also use long division to convert an improper fraction to a proper fraction.

Example 83

Express $\frac{5x^2-71}{(x+5)(x-4)}$ in partial fractions.

Solution

The degree of $5x^2 - 71$ is 2 and the degree of $(x+5)(x-4)$ is also 2. Therefore the quotient is a constant (written as A below) and we assume that

$\frac{5x^2-71}{(x+5)(x-4)} \equiv A + \frac{B}{(x+5)} + \frac{C}{(x-4)}$

Multiplying through by $(x+5)(x-4)$

$\Rightarrow 5x^2 - 71 \equiv A(x+5)(x-4) + B(x-4) + C(x+5)$

Comparing coefficients of;

$x^2; A = 5$

$let\ x = -5; 5(-5)^2 - 71 = B(-5-4)$

$\Rightarrow 54 = -9B \quad \therefore B = -6$

$let\ x = 4;\ \ 5(4)^2 - 71 = C(4+5)$

$\Rightarrow 9 = 9C;\ \therefore C = 1$

Therefore, we have

$\frac{5x^2-71}{(x+5)(x-4)} = 5 - \frac{6}{(x+5)} + \frac{1}{(x-4)}$

Example 84

Express $\frac{3x^4+7x^3+8x^2+53x-186}{(x+4)(x^2+9)}$ in partial fractions.

Solution

The degree of $3x^4 + 7x^3 + 8x^2 + 53x - 186$ is 4 and the degree of $(x+4)(x^2+9)$ is 3, therefore the quotient is a polynomial of degree $(4-3) = 1$ and so we assume that

$\frac{3x^4+7x^3+8x^2+53x-186}{(x+4)(x^2+9)} = Ax + B + \frac{C}{(x+4)} + \frac{Dx+E}{(x^2+9)}$

Multiplying through by $(x+4)(x^2+9)$ gives

$3x^4 + 7x^3 + 8x^2 + 53x - 186 \equiv (Ax+B)(x+4)(x^2+9) + C(x^2+9) + (Dx + E)(x+4)$

Comparing coefficients of the x^4 terms gives $A = 3$.

Comparing coefficients of the x^3 terms gives

$7 = 4A + B$

$\Rightarrow 7 = 4(3) + B;\ \therefore B = -5$

$let\ x = -4;\ \ 3(-4)^4 + 7(-4)^3 + 8(-4)^2 + 53(-4) - 186 = C[(-4)^2 + 9]$

$\Rightarrow 50 = 25C;\ \therefore C = 2$

Comparing coefficients of the x^2 term gives

$8 = 9A + 4B + C + D$

$\Rightarrow 8 = 9(3) + 4(-5) + 2 + D;\ \therefore D = -1$

Comparing the constant terms

$-186 = 36B + 9C + 4E$

$\Rightarrow -186 = 36(-5) + 9(2) + 4E;\ \therefore E = -6$

$\therefore \frac{3x^4+7x^3+8x^2+53x-186}{(x+4)(x^2+9)} = 3x-5+\frac{2}{(x+4)}-\frac{x+6}{(x^2+9)}$

Exercise 4.2d

1. Express the following into partial fractions;

(a) $\frac{x^2}{(x-1)(x-2)} Ans(A=-1, B=4)$

(b) $\frac{x^4+1}{(x-1)(x+1)^2} Ans(A=1/2, B=3/2, C=-1)$

(c) $\frac{2x^2+2x+3}{x^2-1}$

(d) $\frac{x^3+2x^2-2x+2}{(x-1)(x+3)}$

Exercise 4.2e

In the following questions, write each of the expressions in partial fractions.

1. Denominator with two distinct linear factors

(a) $\frac{3x-1}{(x+3)(x-2)}$

(b) $\frac{5x+6}{(x+4(x-3)}$

(c) $\frac{2x+1}{(x+2)(x+1)}$

(d) $\frac{9-8x}{(2x-1)(3-x)}$

2. Denominator with three linear factors

(a) $\frac{2}{(x+3)(x+2)(x+1)}$

(b) $\frac{x^2-9x+2}{(x+1)(x-1)(x-2)}$

(c) $\frac{x+1}{(x+3)(x+2)(x-1)}$

(d) $\frac{2x^2-7x+1}{(2x+1)(2x-1)(x-2)}$

3. Denominator with a quadratic factor

(a) $\frac{5x^2-3x+1}{(x^2+1)(x-2)}$

(b) $\frac{6x+7}{(x^2+2)(x+3)}$

(c) $\frac{6x^2-7x-11}{(x^2+1)(x-5)}$

(d) $\frac{9x+7}{(2x^2+3)(x+2)}$

4. Denominator with repeated factor

(a) $\frac{2x+3}{(x+2)^2}$

(b) $\frac{3x-14}{x^2-8x+16}$

(c) $\frac{5x+7}{(x+1)^2(x+2)}$

(d) $\frac{2x^2+9x+24}{x^3+4x^2-3x-18}$

5. Improper fractions

(a) $\frac{x^2+7x-14}{(x+5)(x-3)}$

(b) $\frac{2x^2+x-5}{(x+2)(x+1)}$

(c) $\frac{x^3+4x^2-x-17}{(x+3)(x-2)}$

(d) $\frac{3x^3-10x^2+2x-1}{(3x-1)(x-3)}$

(e) $\frac{x^3+7x^2-18}{(x+4)(x-1)(x-2)}$

(f) $\frac{x^4+x^3-19x^2-44x-21}{(x+3)(x+2)(x+1)}$

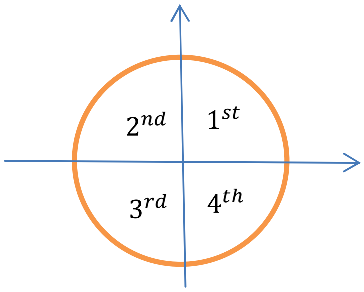
Uses of partial fractions;

1. Partial fractions may be used to simplify the differentiation of a complicated expression.
2. Partial fractions are also used to simplify expressions before using the binomial theorem.
3. By far the most important use of partial fractions is in integration.

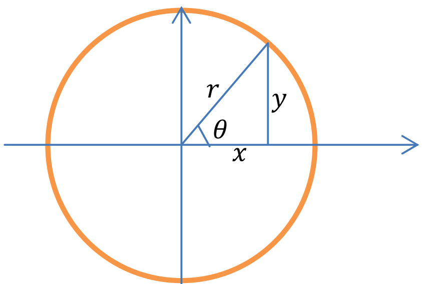
CHAPTER 5
TRIGONOMETRY

5.1 Trigonometric Ratios

- Angles which are measured clockwise are negative and those measured anti-clockwise are positive.
- The axes divide the plane into four quadrants



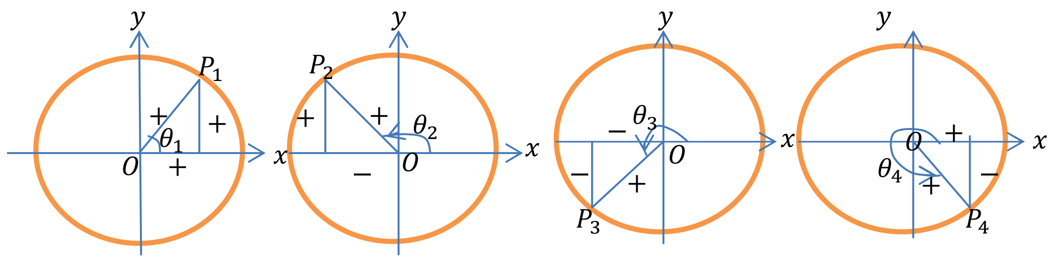
- There are six Trigonometrical ratios as derived from the right angled triangle below;



$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)} = \frac{y}{x}$

$cosec\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

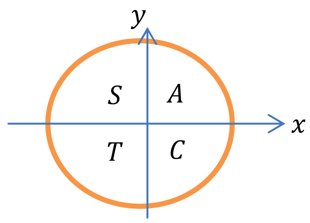
- The signs of Trigonometrical ratios in the various quadrants can be deduced as follows;



The table below gives a summary of what happens in each quadrant.

Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$	Positive ratio	Letter
1st	+	+	+	ALL +	A
2nd	+	−	−	Sine +	S
3rd	−	−	+	Tangent +	T
4th	−	+	−	Cosine +	C

The letters are placed in a unit circle in an anticlockwise direction. And the order can be remembered using the acronym "**All Students Take Care**"



Trigonometrical ratios of special angles

The special angles are $\{0^0, 30^0, 45^0, 60^0\, and\ 90^0\}$

Angle θ^0	0	30	45	60	90
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

Note: The cosine of a negative angle is positive because, cosine is an even function. The sine and tangent of a negative angle are negative because these are odd functions.

i.e.$\cos(-30^0) = +\cos(30^0) = \frac{\sqrt{3}}{2}$

$\sin(-30^0) = -\sin(30^0) = -\frac{1}{2}$ and $\tan(-30^0) = -\tan(30^0) = -\frac{1}{\sqrt{3}}$

Examples 1:

Write each of the following as trigonometric ratios of acute angles.

(a) $\cos 230^0$ (b) $\sin 140^0$

Solution

(a) $\cos 230^0 = -\cos 50^0$ (in the 3rd quadrant)

(b) $\sin 140^0 = +\sin 40^0$ (in the 2nd quadrant)

Example 2

Evaluate the following without using tables of calculator. Leave your answer as surds.

(a) $\cos 135^0$ (b) $\tan 240^0$(c)$\sin(-45^0)$ (d) $\sin 480^0$.

Solution

(a) $\cos 135^0\ (2nd\ quadrant) = -\cos 45^0 = -\frac{1}{\sqrt{2}}$

(b) $\tan 240^0\ (3rd\ quadrant) = +\tan 60^0 = +\sqrt{3}$

(c) $\sin(-45^0)\ (4th\ quadrant) = -\sin 45^0 = -\frac{1}{\sqrt{2}}$

(d) $\sin 480^0$in this case the rotating arm will be in the same position as for 120^0.

$\sin 480^0 = \sin 120^0\ (2nd\ quadrant) = +\sin 60^0 = +\frac{\sqrt{3}}{2}$

Note: The Trigonometric ratio of any angle θ greater than 360^0 is the same as the trigonometric ratio of $(\theta - 360^0)$.

Maximum and minimum values of *Sine*and *Cosine*.

The trigonometric ratios of all angles differ from the trigonometric ratios of acute angles only in sign.

From the definition of the sine an cosine:

The maximum value of $\sin\theta$ is +1 (when $\theta = 90^0, 450^0, \dots$)

The minimum value of $\sin\theta$ is -1 (when $\theta = 270^0, 630^0, \dots$)

The maximum value of $\cos\theta$ is +1 (when $\theta = 0^0, 360^0, \dots$)

The minimum value of $\cos\theta$ is -1 (when $\theta = 180^0, 540^0, \dots$)

Example 3

Write down the maximum and minimum values of each of the following and state the smallest value of θ from $0^0\ to\ 360^0$for which these values occur.

(a) $1 - 2\cos\theta$ (b) $3\sin\theta - 1$

Solution

(a) The maximum value of $1 - 2\cos\theta$ is $1 - -2 = 3$ and occurs when $\cos\theta = -1$ i.e. $\theta = 180^0$.

The minimum value of $1 - 2\cos\theta$ is $1 - 2 = -1$ and occurs when $\cos\theta = 1$ i.e. $\theta = 0^0$.

(b) The maximum value of $3\sin\theta - 1$is $3(1) - 1 = 2$ and occurs when $\sin\theta = 1$ i.e. $\theta = 90^0$.

The minimum value of $3\sin\theta - 1$is $3(-) - 1 = -4$ and occurs when $\sin\theta = -1$ i.e. $\theta = 270^0$.

Graphs of trigonometric functions

Qn.1 On separate graphs draw the graphs of;

(i) $y = \sin\theta$ (ii) $y = \cos\theta$ (iii) $y = \tan\theta$ using the interval of 30^0, from for the range $0^0 \leq \theta \leq 360^0$.

Qn.2 (a) Plot the graph of $y = 1 + \sin 2x$for $0^0 \leq x \leq 180^0$.

(b) Obtain approximate solutions from the graph to the equations

(i) $1 + \sin 2x = 1.2 Ans(x = 6^0\ or\ 84^0)$

(ii) $\sin 2x\ Ans(x = 22^0\ or\ 68^0)$

Solving trigonometric equations:

Example 4

Solve the equation $\cos x = \frac{\sqrt{3}}{2}$ for values of x such that $0^0 \leq \theta \leq 360^0$

Solution

$x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^0, 330^0$

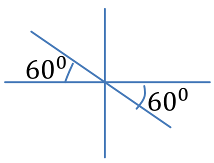
Example 5

Solve the equation $\tan x = -\sqrt{3}$ for values of x such that $-180^0 \leq x \leq 180^0$.

Solution

We first ignore the acute angle and find the acute angle with a tangent of $\sqrt{3}$;

This is 60^0 so the solutions will make an angle of 60^0 with the x-axis and they will be in the 2nd and the 4th quadrants.



For the range $-180^0 \leq x \leq 180^0, x = -60^0\ or\ 120^0$.

Example 6

Solve the following equations for $0^0 \leq x \leq 360^0$

(a) $\cos x = 0.2$ (b) $\sin x = -0.2$

Solution

(a) $x = \cos^{-1} 0.2 = 78.46^0, 281.54^0$

(b) Ignoring the negative sign first, the angle with sine 0.2 is 11.54^0.
Sine is negative in the 3rd and the 4th quadrants
For the range for $0^0 \leq x \leq 360^0, x = 191.54^0, 348.46^0$

Example 7

Solve $\cos x = -0.3$for $-180^0 \leq x \leq 180^0$.
Solution
Ignoring the minus sign
$\cos^{-1} 0.3 = 72.54^0$from the calculator or table.
Because cosine is negative in the 2nd and 3rd quadrants then
For the range $-180^0 \leq x \leq 180^0$ $x = 107.46^0$ or -107.46^0

Example 8

Solve $\sin(x + 10^0) = -0.5$ for $0^0 \leq x \leq 360^0$
Solution
Ignoring the minus sign $\sin^{-1} 0.5 = 30^0$. But sine is negative in the 3rd and 4th quadrants
$\Longrightarrow (x + 10^0) = 210^0, x = 200^0or(x + 10^0) = 330^0, x = 320^0$
$\therefore$ For the range $0^0 \leq x \leq 360^0, x = 200^0, 320^0$

Example 9

Solve $\cos 2x = 0.6$ for $0^0 \leq x \leq 360^0$
Solution
$2x = \cos^{-1} 0.6$
$\Longrightarrow 2x = 53.13^0, 306.87^0, 413.13^0, 666.87^0$
$\therefore x = 26.57^0, 153.43^0, 206.57^0, 333.43^0$for the range $0^0 \leq x \leq 360^0$

Example 10

Solve $3(\tan x + 1) = 2$ for $-180^0 \leq x \leq 180^0$
Solution
$\Longrightarrow 3\tan x + 3 = 2$
$\Longrightarrow \tan x = -1/3$
Ignoring the minus sign give $x = \tan^{-1}(1/3) = 18.43^0$.
But tangent is negative in the 2nd and 4th quadrants
$\therefore$ For the range $-180^0 \leq x \leq 180^0, x = -18.43^0$ or 161.57^0

Example 11

Solve $sin^2 x + \sin x \cos x = 0$ for $0^0 \leq x \leq 360^0$
$$\Longrightarrow \sin x\,(\sin x + \cos x) = 0$$
Either $\sin x = 0, , x = 0^0, 180^0, 360^0$
or$\sin x + \cos x = 0$

$\Rightarrow \sin x = -\cos x$

$\Rightarrow \tan x = -1$ (dividing through by $\cos x$)

Ignoring the minus sign $\tan^{-1} 1 = 45^0$. But tangent is negative in the 2nd and 4th quadrants

$\Rightarrow x = 135^0, 315^0$

For the range $0^0 \leq x \leq 360^0, x = 0^0, 135^0, 180^0, 315^0, 360^0$.

Note*: It is important to factorise out* $\sin x$*, and do not attempt to cancel by* $\sin x$*, cancelling would lead to* $sinx = -cosx$ *and so the solution arising from* $\sin x$ *would be lost.*

Example 12

Solve $6cos^2x - cosx - 1 = 0$ for $0^0 \leq x \leq 360^0$

Solution

Factorising $(3cosx + 1)(2cosx - 1) = 0$

Either $3osx + 1 = 0$

$\Rightarrow \cos x = -\frac{1}{3}$

Now $\cos^{-1}\frac{1}{3} = 70.53^0$, but cosine is negative in the 2nd and 3rd quadrants

$x = 109.47^0,\ \ 250.53^0$

$or\ 2cosx - 1 = 0$

$\Rightarrow cosx = \frac{1}{2}, x = 60^0, 300^0$

$\therefore$ For the range $0^0 \leq x \leq 360^0, x = 60^0, 109.47^0, 250.53^0, 300^0$

Exercise 5.1a

1. Express in terms of the trigonometric ratios of acute angles

(a) $\sin 170^0$ (b) $\sin 250^0$ (c) $\cos 200^0$ (d) $\cot 156^0$ (e) $\tan(-50)^0$

$Ans((a)sin10^0(b){-sin}70^0(c){-cos}20^0(d){-cot}24^0(e){-tan}50^0)$

2. Write down the values of the following leaving surds in your answers.

(a) $\cos 270^0$ (b) $\tan 135^0$ (c) $\sin(-120^0)$ (d) $\sin 540^0$

$Ans((a)0\ (b)-1\ (c)-\sqrt{3}/2\ \ (d)0$

3. Find the values of θ from $180^0$and 360^0, inclusive, which satisfy the following equations.

(a) $\cos\theta = -1/2$ (b) $\sin\theta = -0.7660$ (c) $\sin(\theta - 30^0) = -\sqrt{3}/2$

4. Solve the following equations for $0^0 \leq \theta \leq 360^0$

(a) $sin^2\theta = 0.25$ (b) $\cos 3\theta = \sqrt{3}/2$ (c) $sec2\theta = 3$ (d) $\tan(2\theta - 45^0) = 1/2$

5. Solve the following equations for values of θ from -180^0 to $+180^0$, inclusive;

(a) $tan^2\theta + tan\theta = 0$ (b) $2sin^2\theta - sin\theta - 1 = 0$ (c) $4cos^3\theta = \cos\theta$

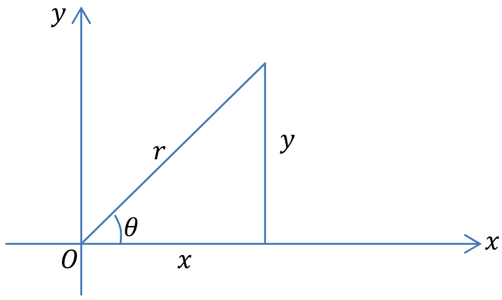
(d) $3cos\theta + 2sec\theta + 7 = 0$ (e) $tan\theta = 4cot\theta + 3$ (f) $5sin\theta + 6cosec\theta = 17$

6. Write down the maximum and minimum values of the following expressions giving the smallest positive or zero value of θ for which they occur.

(a) $-\frac{1}{2}\sin 2\theta$ (b) $3 + 2\cos 3\theta$ (c) $\frac{1}{4-3\cos\theta}$

The Pythagoras theorem

Consider the right angled triangle below



Using Pythagoras theorem, $x^2 + y^2 = r^2$..........(*)

But $\cos\theta = \frac{x}{r}$, $x = r\cos\theta$ and $\sin\theta = \frac{y}{r}$, $y = r\sin\theta$

Equation (*) becomes; $(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

$\therefore \boldsymbol{cos^2\theta + sin^2\theta = 1}$ ....(1)

Dividing equation (1) by $cos^2\theta$

$\Longrightarrow \frac{cos^2\theta}{cos^2\theta} + \frac{sin^2\theta}{cos^2\theta} = \frac{1}{cos^2\theta}$

$\therefore \boldsymbol{1 + tan^2\theta = sec^2\theta}$ ....(2)

Dividing equation (1) by $sin^2\theta$

$\Longrightarrow \frac{cos^2\theta}{sin^2\theta} + \frac{sin^2\theta}{sin^2\theta} = \frac{1}{sin^2\theta}$

$\therefore \boldsymbol{cot^2\theta + 1 = cosec^2\theta}$....................................(3)

Example 13

Solve the equation $1 + \cos\theta = 2sin^2\theta$, for values of θ between 0^0 and 360^0.

Solution

$\Longrightarrow sin^2\theta = 1 - cos^2\theta$ ....(1)

$1 + \cos\theta = 2sin^2\theta$ ....(2)

Putting equation (1) into equation (2)

$\Longrightarrow 1 + cos\theta = 2 - 2cos^2\theta$

$\Longrightarrow 2cos^2\theta + cos\theta - 1 = 0$

$\Longrightarrow (2cos\theta - 1)(cos\theta + 1) = 0$

Either $cos\theta = 1/2$; $\theta = 60^0, 300^0$

Or $cos\theta = -1$; $\theta = 180^0$

Therefore the roots of the equation between 0^0 and 360^0 are 60^0, 180^0 and 300^0.

Example 14

Solve the equation $4cos\theta - 3sec\theta = 2tan\theta$for $-180^0 \leq \theta \leq +180^0$.

Solution

$\Rightarrow 4cos\theta - \frac{3}{cos\theta} = 2\frac{sin\theta}{cos\theta}$

Multiplying through by $cos\theta$

$\Rightarrow 4cos^2\theta - 3 = 2sin\theta$

$\Rightarrow 4(1 - sin^2\theta) - 3 = 2sin\theta$

$\Rightarrow 4 - 4sin^2\theta - 3 - 2sin\theta = 0$

$\Rightarrow 4sin^2\theta + 2sin\theta - 1 = 0$

$\Rightarrow sin\theta = \frac{-2\pm\sqrt{4+16}}{8}$

Either $sin\theta = 0.3090;\ \theta = 18^0, 162^0$

Or $sin\theta = -0.8090;\ \theta = -54^0, -126^0$

Thus for the range $-180^0 \leq \theta \leq +180^0, \theta = -126^0, -54^0, 18^0, 162^0$

Example 15

Prove the identity $tan^2\theta + sin^2\theta = (sec\theta + cos\theta)(sec\theta - cos\theta)$

Solution

From the RHS

$$\Rightarrow (sec\theta + cos\theta)(sec\theta - cos\theta) = sec^2\theta - sec\theta cos\theta + cos\theta sec\theta - cos^2\theta$$

$$= sec^2\theta - cos^2\theta$$

$$= (1 + tan^2\theta) - (1 - sin^2\theta)$$

$$= tan^2\theta + sin^2\theta$$

$$= LHS$$

$\therefore tan^2\theta + sin^2\theta = (sec\theta + cos\theta)(sec\theta - cos\theta)$................■

Example 16

Prove the identity $\sqrt{\frac{1-cos\theta}{1+cos\theta}} = cosec\theta - cot\theta$

Solution

From LHS

$\Rightarrow \sqrt{\frac{1-cos\theta}{1+cos\theta}} = \sqrt{\left(\frac{1-cos\theta}{1+cos\theta}\right)\left(\frac{1-cos\theta}{1-cos\theta}\right)}$rationalising the denominator

$$= \sqrt{\frac{(1-cos\theta)^2}{1-cos^2\theta}}$$

$$= \sqrt{\frac{(1-cos\theta)^2}{sin^2\theta}}$$

$$= \frac{1-cos\theta}{sin\theta}$$

$$= \frac{1}{sin\theta} - \frac{cos\theta}{sin\theta}$$

$$= cosec\theta - cot\theta$$

$= RHS$

$\therefore \sqrt{\frac{1-cos\theta}{1+cos\theta}} = cosec\theta - cot\theta$ ……………..■

Example 17

If $s = sin\theta$, simplify

(a) $\sqrt{(1-s^2)}$ (b) $\frac{s}{\sqrt{(1-s^2)}}$ (c) $\frac{1-s^2}{s}$

Solution

(a) $\sqrt{(1-s^2)} = \sqrt{(1-sin^2\theta)} = \sqrt{cos^2\theta} = cos\theta$

(b) $\frac{s}{\sqrt{(1-s^2)}} = \frac{sin\theta}{\sqrt{(1-sin^2\theta)}} = \frac{sin\theta}{\sqrt{cos^2\theta}} = \frac{sin\theta}{cos\theta} = tan\theta$

(c) $\frac{1-s^2}{s} = \frac{1-sin^2\theta}{sin\theta} = \frac{cos^2\theta}{sin\theta} = cos\theta cot\theta$

Example 18

Eliminate θ from the equations $x = a\ sin\theta, y = b\ tan\theta$.

Solution

Since $sin\theta$ and $tan\theta$ are the reciprocals of $cosec\theta$ and $cot\theta$ we use the identity $cosec^2\theta = cot^2\theta + 1$

$\Longrightarrow cosec\theta = \frac{a}{x}\ and\ cot\theta = \frac{b}{y}$

Substituting into the identity $cosec^2\theta = cot^2\theta + 1$

$\Longrightarrow \frac{a^2}{x^2} = \frac{b^2}{y^2} + 1$

Exercise 5.1b

1. Solve the equation $2cos^2\theta = 1 + sin\theta$ for $0^0 \leq \theta \leq 360^0 Ans(\theta = 30^0, 150^0, 270^0)$
2. If $x = a\ cos\theta$, simplify (i) $a^2 - x^2$ (ii) $\left(1 - \frac{x^2}{a^2}\right)^{5/2}$
3. If $\sin\theta = \frac{a^2-b^2}{a^2+b^2}$, find the values of $\cos\theta$ and $\tan\theta$.
4. Solve the following equations, giving values of θ from 0^0 to 360^0 inclusive

(a) $3 - 3cos\theta = 2sin^2\theta Ans(0^0, 60^0, 300^0, 360^0)$

(b) $sin^2\theta + sin\theta + 1 = 0 Ans(270^0,)$

(c) $sec^2\theta = 3tan\theta - 1 Ans(45^0, 63.4^0, 225^0, 243.4^0)$

(d) $cosec^2\theta = 3 + cot\theta Ans(26.6^0, 135^0, 206.6^0, 315^0)$

(e) $3tan^2\theta + 5 = 7sec\theta Ans(60^0, 300^0)$

(f) $2cot^2\theta + 8 = 7cosec\theta Ans(30^0, 41.8^0, 138.2^0, 150^0)$

5. Show that $sin^2\theta + (1 + cos\theta)^2 = 2(1 + cos\theta)$
6. If $sin\theta = 1/\sqrt{3}$ and $0^0 < \theta < 360^0$ find the values of the other trigonometric ratios f the angle θ.
7. Eliminate θ from the following equations

(a) $x = a\ cos\theta, y = b\ sin\theta$

(b) $x = a\ cot\theta, y = b\ cosec\theta$

(c) $x = a\ tan\theta, y = b\ cos\theta$

(d) $x = 1 - sin\theta, y = 1 + cos\theta$

(e) $x = a\ sec\theta, y = b + c\ cos\theta$

(f) $x = sin\theta + cos\theta, y = sin\theta - cos\theta$

(g) $x = sec\theta + tan\theta, y = sec\theta - tan\theta$

8. If $tan\theta + sin\theta = x\ and\ tan\theta - sin\theta = y$, prove that $(x^2 - y^2)^2 = 16xy$
9. Show that $\left(\frac{1+sinx}{1+cosx}\right)\left(\frac{1+secx}{1+cosecx}\right) = tanx$
10. Show that $\frac{sin\phi}{1-cos\phi} = \frac{1+cos\phi}{sin\phi}$
11. If $6cos^2\theta + 2sin^2\theta = 5$, show that $tan^2\theta = \frac{1}{3}$
12. Show that;

(a) $tan\theta + cot\theta = sec\theta cosec\theta$

(b) (b) $4 - 3cos^2\theta = 3sin^2\theta + 1$

(c) $\frac{1+cos\theta}{1-cos\theta} \cdot \frac{sec\theta-1}{sec\theta+1} = 1$

(d) $\frac{1-sin\theta}{1+sin\theta} = (sec\theta - tan\theta)^2$

13. If $x' = xcos\theta + ysin\theta\ and\ y' = xsin\theta - ycos\theta$ show that $(x')^2 + (y')^2 = x^2 + y^2$
14. Prove that $\frac{sin\theta}{1+cos\theta} + \frac{1+cos\theta}{sin\theta} = \frac{2}{sin\theta}$
15. Prove that $\sin 330^0 \cos 390^0 - \cos 570^0 \sin 510^0 = 0$
16. If $sec\theta - cos\theta = a$ and $cosec\theta - sin\theta = b$, prove that $a^2b^2(a^2 + b^2 + 3) = 1$
17. Prove that $\frac{cot\alpha+tan\beta}{cot\beta+tan\alpha} = cot\alpha tan\beta$
18. Prove that $\frac{1+sin\theta}{cos\theta} = \frac{cos\theta}{1-sin\theta} = sec\theta + tan\theta$
19. Show that $\frac{cos\theta-1}{sec\theta+tan\theta} + \frac{cos\theta+1}{sec\theta-tan\theta} = 2(1 + tan\theta)$
20. If $xcos\theta + ysin\theta = a\ and\ xsin\theta - ycos\theta = b$ prove that

(a) $tan\theta = \frac{bx+ay}{ax-by}$ (b) $x^2 + y^2 = a^2 + b^2$

5.2 Compound Angle Formulae

The formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ *and* $\tan(A \pm B)$

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

2. $\sin(A - B) = \sin A \cos B - \cos A \sin B$
3. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
4. $\cos(A - B) = \cos A \cos B + \sin A \sin B$
5. $\tan(A + B) = \frac{\sin(A+B)}{\cos(A+B)}$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$ Dividing through by $\cos A \cos B$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

6. $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Example 19

Evaluate the following without the use of table or a calculator

(a) $\cos 75^0$ (b) $\tan 15^0$

Solution

(a) $\cos 75^0 = \cos(30^0 + 45^0)$

$$= \cos 30^0 \cos 45^0 - \sin 30^0 \sin 45^0$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

(b) $\tan 15^0 = \tan(45^0 - 30^0)$

$$= \frac{\tan 45^0 - \tan 30^0}{1 + \tan 45^0 \tan 30^0}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \times \frac{\sqrt{3}}{3}} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

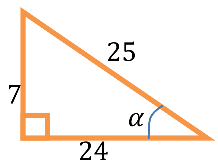
Example 20

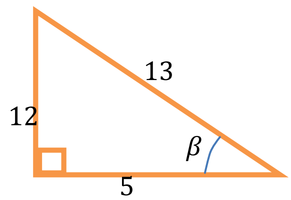
Given that α and β are acute angles with $\sin \alpha = 7/25$and $\cos \beta = 5/13$, find

(a) $\sin(\alpha + \beta)$

(b) $\tan(\alpha + \beta)$

Solution





Since $\sin \alpha = 7/25$, $\cos \alpha = 24/25$, $\tan \alpha = 7/24$ and

Since $\cos \beta = 5/13$,$\sin \beta = 12/13$, $\tan \beta = 12/5$

(a) $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$= \frac{7}{25} \times \frac{5}{13} + \frac{24}{25} \times \frac{12}{13} = \frac{323}{325}$$

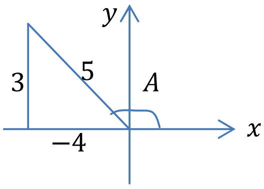
(b) $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

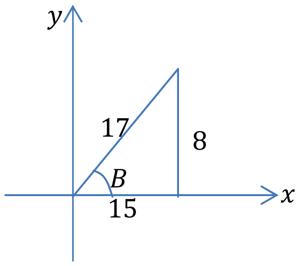
$$= \frac{\frac{7}{24} + \frac{12}{5}}{1 - \left(\frac{7}{24}\right)\left(\frac{12}{5}\right)} = \frac{323}{36}$$

Example 21

If $\sin A = 3/5$and $\cos B = 15/17$, where A is obtuse and B is cute, find the exact value of $\sin(A + B)$

Solution





Since $\sin A = 3/5$, $\cos A = -4/5$

Since $\cos B = 15/17$, $\sin B = 8/17$

$\Longrightarrow \sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \left(\frac{3}{5}\right) \times \left(\frac{15}{17}\right) + \left(\frac{-4}{5}\right) \times \left(\frac{8}{17}\right)$$

$$= \frac{45}{85} - \frac{32}{85}$$

$$= \frac{13}{85}$$

Example 22

Solve the equation $\cos\theta \cos 20^0 + \sin\theta \sin 20^0 = 0.75$ for $0^0 \leq \theta \leq 360^0$

Solution

$\Longrightarrow \cos(\theta - 20^0) = 0.75$

$\Longrightarrow \theta - 20^0 = \cos^{-1} 0.75$

$\Longrightarrow \theta - 20^0 = 41.41^0, 318.59^0$

$\therefore \theta = 61.41^0, 338.59^0$ for the range $0^0 \leq \theta \leq 360^0$

Example 23

If $\sin(x + \alpha) = \cos(x - \beta)$, find $\tan x$ in terms of α and β

Solution

$\Longrightarrow \sin x \cos\alpha + \cos x \sin\alpha = \cos x \cos\beta + \sin x \sin\beta$

$\Longrightarrow \sin x \cos\alpha - \sin x \sin\beta = \cos x \cos\beta - \cos x \sin\alpha$

$\Longrightarrow \sin x (\cos\alpha - \sin\beta) = \cos x (\cos\beta - \sin\alpha)$

$\Longrightarrow \frac{sinx}{cosx} = \frac{(\cos\beta - \sin\alpha)}{(\cos\alpha - \sin\beta)}$

$\therefore \tan x = \frac{(\cos\beta - \sin\alpha)}{(\cos\alpha - \sin\beta)}$ ....■

Double angle formulae

From the identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$by putting $B = A$

$\Longrightarrow \sin(A+A) = \sin A\cos A + \cos A\sin A$

$\therefore \boldsymbol{\sin 2A = 2\sin A\cos A}$

Similarly by using $\cos(A+B) = \cos A\cos B - \sin A\sin B$ by putting $B = A$

$\Longrightarrow \cos(A+A) = \cos A\cos A - \sin A\sin A$

$\therefore \boldsymbol{\cos 2A = cos^2 A - sin^2 A}$

Using the identity of $\cos 2A$

$\Longrightarrow \cos 2A = 1 - sin^2A - sin^2A = \boldsymbol{1 - 2sin^2A}$

Also $\cos 2A = cos^2A - (1 - cos^2A) = \boldsymbol{2cos^2A - 1}$

From the identity $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$ by putting $B = A$

$\Longrightarrow \tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A\tan A}$

$\therefore \boldsymbol{\tan 2A = \frac{2\tan A}{1 - tan^2A}}$

Half -angle formulae

From the double-angle formula for the cosine, it follows that

$\cos\theta = 2cos^2\frac{\theta}{2} - 1$

Hence $2cos^2\frac{\theta}{2} = 1 + \cos\theta$

$\therefore \boldsymbol{cos^2\frac{\theta}{2} = \frac{1}{2}(1 + \cos\theta)}$

And similarly, as

$\cos\theta = 1 - 2sin^2\frac{\theta}{2}$

Hence $2sin^2\frac{\theta}{2} = 1 - \cos\theta$

$\therefore \boldsymbol{sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta)}$

Hence given the value of $\cos\theta$ we can find $\sin\frac{\theta}{2}$ *and* $\cos\frac{\theta}{2}$

Example 24

Solve the equation $3\cos 2\theta + \sin\theta = 1$ for values of θ from 0^0 to 360^0 inclusive.

Solution

$\Longrightarrow 3(1 - 2sin^2\theta) + \sin\theta = 1$

$\Longrightarrow 3 - 6sin^2\theta + \sin\theta = 1$

$\Longrightarrow 6sin^2\theta + \sin\theta - 2 = 0$

$\Longrightarrow (3\sin\theta - 2)(2\sin\theta + 1) = 0$

$\Longrightarrow \sin\theta = \frac{2}{3}, \theta = 41.8^0, 138.2^0 \text{or} \sin\theta = -\frac{1}{2}, \theta = 210^0, 330^0$

$\therefore$ For the range $0^0 \leq \theta \leq 360^0, \theta = 41.8^0, 138.2^0, 210^0, 330^0$.

Example 25

Prove that $\sin 3A = 3\sin A - 4\sin^3 A$

Solution

$\Longrightarrow \sin(2A + A) = \sin 2A\cos A + \cos 2A\sin A$

$\Longrightarrow \sin 3A = 2\sin A\cos A\cos A + (1 - 2sin^2A)\sin A$

$= 2\sin A\, cos^2A + \sin A - 2sin^3A$

$= 2\sin A\,(1 - sin^2A) + \sin A - 2sin^3A$

$= 2\sin A\ - 2sin^3A + \sin A - 2sin^3A$

$= 3\sin A - 4sin^3A$

$\therefore \sin 3A = 3\sin A - 4\sin^3 A$ ……………..■

Qn. Prove that $\cos 3A = 4\cos^3 A - 3\cos A$

Example 26

Prove that (a)$\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$ (b) $cosec2\theta + cot2\theta = \cot\theta$

Solution

(a) From the LHS

$\Longrightarrow \frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+2cos^2\theta-1}$

$= \frac{2\sin\theta\cos\theta}{2cos^2\theta}$

$= \frac{\sin\theta}{\cos\theta}$

$= \tan\theta$

$= RHS$

$\therefore \frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$ …………………..■

(b) From LHS

$\Longrightarrow cosec2\theta + cot2\theta = \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta}$

$= \frac{1+\cos 2\theta}{\sin 2\theta}$

$= \frac{1+2cos^2\theta-1}{2\sin\theta\cos\theta}$

$= \frac{2cos^2\theta}{2\sin\theta\cos\theta}$

$= \frac{\cos\theta}{\sin\theta}$

$= \cot\theta$

$= LHS$

$\therefore\ cosec2\theta + cot2\theta = \cot\theta$ ………………..■

Exercise 5.2a

1. Evaluate

(a) $\sin 50^0\cos 40^0 + \cos 50^0\sin 40^0$ $Ans(1)$

(b) $\cos 75^0\cos 15^0 + \sin 75^0\sin 15^0$ $Ans(1/2)$

2. Without using tables or calculator find the following, leaving surds in your answer

(a) $\sin(45^0 + 30^0)$ $Ans\left(\frac{\sqrt{2}}{4}(\sqrt{3}+1)\right)$

(b) $\sin 15^0$ $Ans\left(\frac{\sqrt{2}}{4}(\sqrt{3}-1)\right)$

(c) $\tan 105^0$ $Ans(-2-\sqrt{3})$

3. Find (i) the greatest and (ii) the least values that each of the following expressions can take and state the smallest value of θ from 0^0 to 360^0 for which these values occur.

(a) $\sin\theta\cos 20^0 + \cos\theta\sin 20^0$ $Ans(1, 70^0;\ -1, 250^0)$

(b) $\cos\theta\cos 40^0 - \sin\theta\sin 40^0$ $Ans(1, 110^0;\ -1, 290^0)$

4. A and B are acute angle such that $\sin A = \frac{12}{13}$ and $\cos B = \frac{4}{5}$. Without the use of tables or calculator find the value of ;

(a) $\sin(A+B)$ $Ans(33/65)$

(b) $\tan(A-B)$ $Ans(33/56)$

5. Show that $\cos(90^0 + A) = -\sin A$

6. C and D are both obtuse angles such that $\sin C = \frac{3}{5}$ and $\sin D = \frac{5}{13}$ Without the use of tables or a calculator, find the values of;

(a) $\sin(C+D)$ $Ans(-56/65)$

(b) $\cos(C-D)$ $Ans(63/65)$

7. Solve the following equations for $0^0 \le \theta \le 360^0$

(a) $\sin\theta\cos 10^0 + \cos\theta\sin 10^0 = -0.5 Ans(200^0, 320^0)$

(b) $\cos 40^0\cos\theta - \sin 40^0\sin\theta = 0.4 Ans(26.4^0, 253.6^0)$

(c) $\sin(\theta + 45^0) = \sqrt{2}\cos\theta$ $Ans(45^0, 225^0)$

8. Prove the following identities;

(a) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(b) $\frac{\sin(A-B)}{\cos A\cos B} + \frac{\sin(B-C)}{\cos B\cos C} + \frac{\sin(C-A)}{\cos C\cos A} = 0$

9. If $\sin C = 1/\sqrt{5}$ find $\sin 2C$, $\cos 2C$ and $\tan 2C$ if

(a) C is acute (b) C is obtuse

10. Eliminate θ from each of the following pairs of simultaneous equations.

(a) $x + 1 = \cos 2\theta$, $y = \sin\theta$

(b) $x = \cos 2\theta$, $y = \cos\theta - 1$

11. Prove the following identities

(a) $cos^2 A + \cos 2A = 2 - 3sin^2 A$

(b) $\frac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$

(c) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

(d) $\frac{1+\tan^2 A}{1-\tan^2 A} = \sec 2A$

(e) $\sin\frac{A}{2} = \frac{\sin A}{1+\cos A}$

(f) $\tan(45^0 + A) = \frac{1+\tan A}{1-\tan A}$

(g) $\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}$

(h) $(\cos\theta + \cos\phi)^2 + (\sin\theta + \sin\phi)^2 = 2 + 2\cos(\theta + \phi)$

12. If $\tan(x+y) = \frac{4}{3}$ and $\tan x = \frac{1}{2}$, evaluate $\tan y$.

13. If $\tan\lambda = \mu$ show that $\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \tan(\theta + \lambda)$

14. Express in terms of the cosines and sines of A, B and C

(i) $\sin(A+B+C)$ (ii) $\cos(A+B+C)$

15. Show that $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

16. If $\cos 30^0 = \sqrt{3}/2$ show that $\sin 15^0 = \frac{\sqrt{(2-\sqrt{3})}}{2}$

17. Show that $\tan 22\frac{1}{2}^0 = \sqrt{2} - 1$ without using tables.

18. If $\tan^2 x = 1 + 2\tan^2 y$ show that $\cos 2x + \sin^2 y = 0$.

19. Prove that $\frac{\cos 3\theta}{\cos\theta} - \frac{\cos 6\theta}{\cos 2\theta} = 2(\cos 2\theta - \cos 4\theta)$

20. If $\sin 3\theta = p$ and $sin^2\theta = \frac{3}{4} - q$, prove that $p^2 + 16q^3 = 12q^2$

21. Given that $\tan A = \frac{3+4x}{4-3x}$ and $\tan B = \frac{6+7x}{7-6x}$, show h=that $\tan(A-B)$ is independent of x.

Inverse trigonometric ratios

This deals with finding the unknowns and simplifying given expressions:

Example 27

Without the use of the mathematical tables or a calculator, find $\tan\theta$ if $\theta = \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{7}{24}$.

Solution

Let $A = \tan^{-1}\frac{5}{12}$; $\tan A = \frac{5}{12}$

And $B = \tan^{-1}\frac{7}{24}$; $\tan B = \frac{7}{24}$

$\Longrightarrow \theta = A + B$

$\Longrightarrow \tan\theta = \tan(A+B)$

$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{\left(\frac{5}{12}\right) + \left(\frac{7}{24}\right)}{1 - \left(\frac{5}{12}\right)\left(\frac{7}{24}\right)}$

$= \frac{204}{253}$

$\therefore \tan\theta = \frac{204}{253}$

Example 28

Without the use of mathematical tables, or calculator, evaluate

$\tan^{-1}\frac{\sqrt{3}}{2}+\tan^{-1}\frac{\sqrt{3}}{5}$, leaving your answer in terms of π.

Solution

Let $A=\tan^{-1}\frac{\sqrt{3}}{2}$; $\tan A=\frac{\sqrt{3}}{2}$

And $B=\tan^{-1}\frac{\sqrt{3}}{2}$; $\tan B=\frac{\sqrt{3}}{2}$

$\Longrightarrow \tan^{-1}\frac{\sqrt{3}}{2}+\tan^{-1}\frac{\sqrt{3}}{5}=A+B$

$\Longrightarrow \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

$=\frac{\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{5}}{1-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{5}\right)}$

$=\sqrt{3}$

$A+B=\frac{\pi}{3}$

$\therefore \tan^{-1}\frac{\sqrt{3}}{2}+\tan^{-1}\frac{\sqrt{3}}{5}=\frac{\pi}{3}$

Example 29

Find the positive value of x that satisfies the equation $\tan^{-1}3x+\tan^{-1}x=\frac{\pi}{4}$, giving your answer correct to 3 decimal places.

Solution

Let $A=\tan^{-1}3x$; $\tan A=3x$

And $B=\tan^{-1}x$; $\tan B=x$

$\Longrightarrow A+B=\frac{\pi}{4}$

$\Longrightarrow \tan(A+B)=\tan\frac{\pi}{4}=1$

$\Longrightarrow \frac{\tan A+\tan B}{1-\tan A\tan B}=1$

$\Longrightarrow \frac{3x+x}{1-(3x)(x)}=1$

$\Longrightarrow 3x^2+4x-1=0$

$\Longrightarrow x=\frac{-(4)\pm\sqrt{(4)^2-4(3)(-1)}}{2(3)}$

$x=0.215$ or $x=-1.55$

$\therefore x=0.215$

Example 30

Solve the equation $\cos^{-1}x+\cos^{-1}(x\sqrt{3})=\frac{\pi}{2}$

Solution

Let $A=\cos^{-1}x$; $\cos A=x, \sin A=\sqrt{1-x^2}$

Let $B=\cos^{-1}(x\sqrt{3})$, $\cos B=x\sqrt{3}$, $\sin B=\sqrt{1-3x^2}$

$A+B=\frac{\pi}{2}$

$\Longrightarrow \cos(A+B) = \cos\frac{\pi}{2} = 0$

$\Longrightarrow \cos A \cos B - \sin A \sin B = 0$

$\Longrightarrow (x)(x\sqrt{3}) - (\sqrt{1-x^2})(\sqrt{1-3x^2}) = 0$

$\Longrightarrow x^2\sqrt{3} = \sqrt{\{1-4x^2+3x^4\}}$

Squaring both sides;

$\Longrightarrow 3x^4 = 1 - 4x^2 + 3x^4$

$\Longrightarrow 4x^2 = 1$

$x = \pm 1/2$

$\therefore x = 1/2$ is the only solution.

Exercise 5.2b

1. Without using a calculator find;

(a) $\sin\theta\ if\ \theta = \sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) Ans(56/65)$

(b) $\cos\theta\ if\ \cos^{-1}\left(\frac{5}{13}\right) - \cos\left(\frac{8}{17}\right) Ans(220/221)$

(c) $\tan\theta\ if\ \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{7}{24}\right) Ans(4/3)$

(d) $\sin\theta\ if\ \sin^{-1}\left(\frac{4}{5}\right) - \cos^{-1}\left(\frac{8}{17}\right) Ans(-13/85)$

(e) $\cos\theta\ if\ \cos^{-1}\left(\frac{24}{25}\right) - \sin^{-1}\left(\frac{15}{17}\right) Ans(297/425)$

2. Evaluate the following. (Give your answers in terms of π and do not use a calculator)

(a) $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{1}{3}\right) Ans(\frac{\pi}{2})$

(b) $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) Ans(\pi/4)$

3. Prove that $\sin(2\sin^{-1}x + \cos^{-1}x) = \sqrt{1-x^2}$
4. Without the use of a calculator show that $2\sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{35}{37}\right) = \sin^{-1}\left(\frac{756}{925}\right)$
5. Find positive value of x that satisfies the equation $\tan^{-1}3x + \tan^{-1}2x = \frac{\pi}{4} Ans(1/6)$
6. Find correct to three decimal places, a positive value of x that satisfies the equation $\tan^{-1}x + \tan^{-1}2x = \tan^{-1}2\ Ans\ (0.425)$
7. Solve the equation $\cos^{-1}x + \cos^{-1}(x\sqrt{8}) = \frac{\pi}{2}\quad Ans(1/3)$
8. Solve the equation $2\sin^{-1}(x\sqrt{6}) + \sin^{-1}(4x) = \frac{\pi}{2}\quad Ans(1/16)$
9. Solve the equation $2\sin^{-1}\left(\frac{x}{2}\right) + \sin^{-1}(x\sqrt{2}) = \frac{\pi}{2}\quad Ans\left\{\sqrt{(6-4\sqrt{2})}\right\}$
10. Evaluate $\sec^{-1}\{2\}$ and $\cot^{-1}\sqrt{3}$
11. Prove that $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
12. Prove that $2\sin^{-1}\left(\frac{1}{2}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
13. Prove that $\cot^{-1}\frac{1}{3} = \cot^{-1}3 + \cos^{-1}\frac{3}{5}$
14. Show that there is a positive value of x which satisfies the equation

$\tan^{-1}(2x+1) + \tan^{-1}(2x-1) = \tan^{-1} 2$

5.3 The factor formulae

- The sums and differences of sins and cosines may be expressed as products of sines and cosines and vice-versa.

$\sin(A+B) = \sin A\cos B + \cos A\sin B$ …………………(1)

$\sin(A-B) = \sin A\cos B - \cos A\sin B$ ………………….(2)

Adding (1) and (2)

$\Rightarrow \sin(A+B) + \sin(A-B) = 2\sin A\cos B$…………………….(3)

We let $A+B = P$ and $A-B = Q$; $A = \frac{P+Q}{2}, B = \frac{P-Q}{2}$.

$\Rightarrow \sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$……………………………….(i)

Subtracting (1) and (2)

$\Rightarrow \sin(A+B) - \sin(A-B) = 2\cos A\sin B$

$\Rightarrow \sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$……………………………….(ii)

$\cos(A+B) = \cos A\cos B - \sin A\sin B$ ………………...(4)

$\cos(A-B) = \cos A\cos B + \sin A\sin B$ …………………(5)

Adding (4) and (5)

$\Rightarrow \cos(A+B) + \cos(A-B) = 2\cos A\cos B$ …………………(6)

$\Rightarrow \cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$ ………………………..(iii)

Subtracting (4) and (5)

$\Rightarrow \cos(A+B) - \cos(A-B) = -2\sin A\sin B$……………….(7)

$\Rightarrow \cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$ …………………………..(iv)

The following is a summary of the factor formulae:

1. $\mathbf{\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}}$
2. $\mathbf{\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}}$
3. $\mathbf{\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}}$
4. $\mathbf{\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}}$

Example 31

Solve the equation $\cos(x+20^0) - \cos(x+80^0) = 0.5$ for $0^0 \le x \le 360^0$

Solution

$\Rightarrow -2\sin(x+50^0)\sin(-30^0) = 0.5$

But $\sin(-30^0) = -\sin 30^0 = -1/2$

$\Rightarrow \sin(x+50^0) = 0.5$

$\Rightarrow (x+50^0) = 30^0, 150^0, 390^0, 510^0$

$\Rightarrow x = -20^0, 100^0, 340^0$

Therefore the roots of the equation between 0^0 and 360^0 are 100^0 and 340^0

Example 32

Prove the identity $\frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A$

Solution

From the LHS

$$\Rightarrow \frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \frac{2\cos\frac{5A+3A}{2}\sin\frac{5A-3A}{2}}{2\cos\frac{3A+5A}{2}\cos\frac{3A-5A}{2}}$$

$$= \frac{2\cos 4A \sin A}{2\cos 4A \cos(-A)}$$

$$= \frac{\sin A}{\cos A} \text{since} \cos(-A) = \cos A$$

$$= \tan A$$

$$= RHS$$

$\therefore \frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A$ ………………..■

Note: It is also useful to be able to use these four standard results the other way round, i.e. to express products as sums and differences.

Example 33

Solve the equation $\sin(x + 15^0)\cos(x - 15^0) = 0.5$, for values of x from 0^0 to 360^0 inclusive.

Solution

$\sin(x + 15^0)\cos(x - 15^0) = 0.5$

$\Rightarrow 2\sin(x + 15^0)\cos(x - 15^0) = 1$

$\Rightarrow \sin 2x + \sin 30^0 = 1$

$\Rightarrow \sin 2x = 1 - \sin 30^0$

$\Rightarrow \sin 2x = 0.5$

$\Rightarrow 2x = 30^0, 150^0, 390^0, 510^0$

$\therefore x = 15^0, 75^0, 195^0, 255^0$ for the range $0^0 \leq x \leq 360^0$

Example 34

Prove $\frac{\sin 8\theta \cos\theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta} = \tan 2\theta$

Solution

From the LHS

$$\Rightarrow \frac{\sin 8\theta \cos\theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta} = \frac{\frac{1}{2}(\sin 9\theta + \sin 7\theta) - \frac{1}{2}(\sin 9\theta + \sin 3\theta)}{\frac{1}{2}(\cos 3\theta + \cos\theta) - \frac{1}{2}(\cos\theta - \cos 7\theta)}$$

$$= \frac{\sin 7\theta - \sin 3\theta}{\cos 3\theta + \cos 7\theta}$$

$$= \frac{2\cos 5\theta \sin 2\theta}{2\cos 5\theta \cos 2\theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \tan 2\theta$$

$$= RHS$$

$\therefore \frac{\sin 8\theta \cos\theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta} = \tan 2\theta$

Example 35

Solve the equation $\sin x + \sin 5x = \sin 3x$ for $0^0 \leq x \leq 180^0$

Solution

$\Longrightarrow 2\sin 3x \cos 2x = \sin 3x$

$\Longrightarrow \sin 3x\,(2\cos 2x - 1) = 0$

Either $\sin 3x = 0$,

$\Longrightarrow 3x = 0^0, 180^0, 360^0, 540^0$

$\Longrightarrow x = 0^0, 60^0, 120^0, 180^0$

Or $\cos 2x = 1/2$

$\Longrightarrow 2x = 60^0, 300^0, 420^0$

$\Longrightarrow x = 30^0, 150^0, 210^0$

$\therefore$ For the range $0^0 \leq x \leq 180^0, x = 0^0, 30^0, 60^0, 120^0, 150^0, 180^0$.

Example 36

If A, B and C are angles of a triangle, prove that

$\cos A + \cos B + \cos C - 1 = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Solution

From the LHS

$\Longrightarrow \cos A + \cos B + \cos C - 1 = (\cos A + \cos B) + (\cos C - 1)$

$= \left(2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\right) + (\cos C - 1)$

But since $A + B = 180^0 - C$

Then $\frac{A+B}{2} = 90^0 - \frac{C}{2}$

$\cos\frac{A+B}{2} = \sin\frac{C}{2}$

Seeing this factor in $\sin\frac{C}{2}$

Then $\cos C - 1 = -2sin^2\frac{C}{2}$

$\Longrightarrow \cos A + \cos B + \cos C - 1 = 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 2sin^2\frac{C}{2}$

$$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right]$$

$$= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right]$$
$$= -2\sin\frac{C}{2}\left[\cos\frac{A+B}{2} - \cos\frac{A-B}{2}\right]$$
$$= -2\left[-2\cos\frac{A}{2}\cos\frac{B}{2}\right]\sin\frac{C}{2}$$
$$\therefore \cos A + \cos B + \cos C - 1 = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \ldots\ldots\ldots\ldots\ldots\blacksquare$$

Exercise 5.3

1. Solve the equation $\cos 6x + \cos 4x + \cos 2x = 0$, for values of x from 0^0 to 90^0 inclusive.
2. Prove the identity $\frac{\cos B+\cos C}{\sin B-\sin C} = \cot\frac{B-C}{2}$
3. If $A + B + C = 180^0$, prove that $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$
4. If $\cos A - \cos B = p$and $\sin A - \sin B = q$, express $\cos(A - B)$ and $\sin(A + B)$ in terms of p and q.
5. Show that $\sin 7x + \sin x - 2\sin 2x\sin 3x = 4cos^2 3x\sin x$
6. If $\sin\theta + \sin\phi = a\ and\ \cos\theta + \cos\phi = b$, show that $cos^2\frac{\theta-\phi}{2} = \frac{1}{4}(a^2 + b^2)$
7. Prove that $\frac{\sin 3A\sin 6A+\sin A\sin 2A}{\sin 3A\cos 6A+\sin A\cos 2A} = \tan 5A$
8. Prove that $\cos 10^0\cos 50^0\cos 70^0 = \sqrt{3}/8$
9. Without using tables, prove that $\cos 165^0 + \sin 165^0 = \cos 135^0$
10. Prove that $1 + \cos 56^0 + \cos 58^0 - \cos 66^0 = 4\cos 28^0\cos 29^0\cos 33^0$
11. Prove that $\frac{1-\sin 36^0+\cos 36^0}{1+\sin 36^0+\cos 36^0} = \frac{3\tan 9^0-tan^3 9^0}{1-3tan^2 9^0}$
12. Prove the following identities if A, B and Care to be taken as the angles of a triangle.

(a) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(b) $\tan A + \tan B + \tan C = \tan C\tan B\tan C$

5.4 The Forms $a\cos x \pm b\sin x$

The expression $a\cos x \pm b\sin x$ can be expressed in the forms $R\cos(x \mp \alpha)$ and $a\sin x \pm b\cos x$ can be expressed in the form $R\sin(x \pm \alpha)$. Where α is an acute angle.

Example 37

Express (a) $4\cos x - 5\sin x$ in the form $R\cos(x + \alpha)$,

(b) $2\sin x + 5\cos x$in the form $R\sin(x + \alpha)$.

Solution

(a) Let $4\cos x - 5\sin x \equiv \text{R}\cos(x + \alpha)$

$$4\cos x - 5\sin x \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$$

Equating coefficients of;

$\cos x$; $4 = R\cos\alpha$ …………………..(1)

$\sin x$; $5 = R\sin\alpha$ …………………….(2)

Squaring and adding equations (1) and (2)

$\Rightarrow R^2 = 4^2 + 5^2$; $R = \sqrt{41}$

Equation (2)÷ (1)

$\Rightarrow \tan\alpha = \frac{5}{4}$; $\alpha = 51.34^0$

$\therefore \mathbf{4\cos x - 5\sin x = \sqrt{41}\cos(x + 51.34^0)}$

(b) Let $2\sin x + 5\cos x \equiv R\sin(x + \alpha)$

$2\sin x + 5\cos x \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$

Equating coefficients of;

$\sin x$; $2 = R\cos\alpha$ …………………(1)

$\cos x$; $5 = R\sin\alpha$ ……………………(2)

Squaring and adding equations (1) and (2)

$\Rightarrow R^2 = 2^2 + 5^2$; $R = \sqrt{29}$

Equation (2)÷(1)

$\Rightarrow \tan\alpha = \frac{5}{2}$; $\alpha = 68.20^0$

$\therefore \mathbf{2\sin x + 5\cos x = \sqrt{29}\sin(x + 68.20^0)}$

Example 38

Solve the equation $5\cos x - 12\sin x = 6.5$for $0^0 \leq x \leq 360^0$.

Solution

Let $5\cos x - 12\sin x \equiv R\cos(x + \alpha)$

$5\cos x - 12\sin x \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$

Equating coefficients of;

$\cos x$; $5 = R\cos\alpha$ …………………..(1)

$\sin x$; $12 = R\sin\alpha$ ……………………..(2)

Squaring and adding equations (1) and (2)

$\Rightarrow R^2 = 5^2 + 12^2$; $R = 13$

Equation (2)÷ (1)

$\Rightarrow \tan\alpha = \frac{12}{5}$; $\alpha = 67.38^0$

$\therefore 5\cos x - 12\sin x = 13\cos(x + 67.38^0)$

$\Rightarrow 13\cos(x + 67.38^0) = 6.5$

$\Rightarrow \cos(x + 67.38^0) = \frac{6.5}{13}$

$\Rightarrow x + 67.38^0 = \cos^{-1} 0.5$

$\Rightarrow x + 67.38^0 = 60^0, 300^0, 420^0$

$\therefore x = 232.62^0, 352.62^0$for the range $0^0 \leq x \leq 360^0$.

Note: when one of the coefficient is a surd e.g. $\sqrt{3}$, $\sqrt{2}$ we strictly use the R-formula only.

Example 39

Solve the equation $\sin x + \sqrt{3}\cos x = 1$ for $0^0 \le x \le 360^0$

Solution

Let $\sin x + \sqrt{3}\cos x \equiv R\sin(x + \beta)$

$\sin x + \sqrt{3}\cos x \equiv R\sin x\cos\beta + \cos x\sin\beta$

Equating coefficients of;

$\sin x$; $1 = R\cos\beta$ …………………(1)

$\cos x$; $\sqrt{3} = R\sin\beta$ ……………………(2)

Squaring and adding equations (1) and (2)

$\Longrightarrow R^2 = 1^2 + (\sqrt{3})^2$; $R = 2$

Equation (2)÷(1)

$\Longrightarrow \tan\beta = \frac{\sqrt{3}}{1}$; $\alpha = 60^0$

$\therefore 2\sin x + \sqrt{3}\cos x \equiv 2\sin(x + 60^0)$

$\Longrightarrow 2\sin(x + 60^0) = 1$

$\Longrightarrow (x + 60^0) = \sin^{-1}(1/2)$

$\Longrightarrow (x + 60^0) = 30^0, 150^0, 390^0$

$x = 90^0, 330^0$for the range $0^0 \le x \le 360^0$.

Example 40

Find the maximum and minimum values of each of the following functions

(a) $f(\theta) = 5\cos\theta + 3\sin\theta$

(b) $f(\theta) = \frac{1}{3+\sin\theta-2\cos\theta}$

Solution

(a) Let $5\cos\theta + 3\sin\theta \equiv R\cos(\theta - \alpha)$

$\equiv R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$\equiv R\cos\theta\cos\alpha + \mathrm{R}\sin\theta\sin\alpha$

Equating the corresponding coefficients gives

$R\cos\alpha = 5$ ………(1)

$R\sin\alpha = 3$ ………….(2)

Squaring each of (1) and (2) and adding give

$R^2cos^2\alpha + R^2sin^2\alpha = 5^2 + 3^2$

$R^2(cos^2\alpha + sin^2\alpha) = 34$

$R^2 = 34$and$R = \sqrt{34}$ (choosing the positive root)

Dividing (2) by (1) give

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{5}$

$\tan\alpha = \frac{3}{5}$; $\alpha = 31.0^0$

Therefore, we have $f(\theta) = \sqrt{34}\cos(\theta - 31.0^0)$

The function f attains its maximum value when the cosine function attains its maximum value. Now $\cos(\theta - 31.0^0)$ has a maximum value of 1 when $\theta - 31.0^0 = 0$. ∴the maximum value of f is $\sqrt{34}$ when $\theta = 31.0^0$. Similarly, the minimum value of f is $-\sqrt{34}$ when $\theta - 31.0^0 = 180^0$; $\theta = 211.0^0$

(b) We start by expressing the denominator of $f(\theta)$ in the form $3 + R\sin(\theta - \alpha)$.

Let $\sin\theta - 2\cos\theta \equiv R\sin(\theta - \alpha)$

$$\equiv R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$

$$\equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

Equating the corresponding coefficients gives

$R\cos\alpha = 1$ ....(1)

$R\sin\alpha = 2$ ....(2)

Squaring each of (1) and (2) and adding give

$R^2cos^2\alpha + R^2sin^2\alpha = 1^2 + 2$

$R^2(cos^2\alpha + sin^2\alpha) = 5$

$R^2 = 5$and$R = \sqrt{5}$ (choosing the positive root)

Dividing (2) by (1) give

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{2}{1}$

$\tan\alpha = \frac{2}{1}$; $\alpha = 63.4^0$

Therefore, we have

$f(\theta) = \frac{1}{3+\sqrt{5}\sin(\theta-63.4^0)}$

The function f attains a minimum when $\sin(\theta - 63.4^0)$ is a maximum. Now $\sin(\theta - 63.4^0)$ has a maximum value of 1 which occurs when $\theta - 63.4^0 = 90^0$

∴ $\theta = 153.4^0$and the minimum point is $\left(153.4^0, \frac{1}{3+\sqrt{5}}\right)$

Similarly, the function f attains a maximum when $\sin(\theta - 63.4^0)$ is a minimum. Now $\sin(\theta - 63.4^0)$ has a minimum value of -1 which occurs when $\theta - 63.4^0 = 270^0$

∴ $\theta = 333.4^0$and the maximum point is $\left(333.4^0, \frac{1}{3-\sqrt{5}}\right)$.

Example 41

For each of

(a) $f(x) = 24\cos\theta + 7\sin\theta$ (b) $f(x) = \frac{1}{2+4\sin\theta-3\cos\theta}$

Find;

(i) the range of values for $f(x)$ (ii) a maximum point (iii) a minimum point.

Solution

(a) (i) Let $24\cos\theta + 7\sin\theta \equiv R\cos(\theta - \alpha)$

$R = \sqrt{24^2 + 7^2} = 25$

$\tan\alpha = \frac{7}{24}$; $\alpha = 16.3^0$

$f(x) = 25\cos(\theta - 16.3^0)$

$the\ range\ is\ from -25\ to +25$i.e.$\boldsymbol{-25 \le f(x) \le 25}$

(ii) For the maximum point, we have $\cos(\theta - 16.3^0) = 1$. Therefore $\theta - 16.3^0 = 0$; $\theta = 16.3^0$

Hence the maximum point is $\left(\boldsymbol{16.3^0, 25}\right)$

(iii) For the minimum point, we have $\cos(\theta - \alpha) = -1$. Therefore

$\theta - 16.3^0 = 180^0$; $\theta = 196.3^0$

Hence the maximum point is $\left(\boldsymbol{196.3^0, -25}\right)$

(b) $f(x) = \frac{1}{2+4\sin\theta - 3\cos\theta}$

Let $4\sin\theta - 3\cos\theta \equiv R\sin(\theta - \alpha)$

$R = \sqrt{4^2 + 3^2} = 5$

$\tan\alpha = \frac{3}{4}$; $\alpha = 36.87^0$

$\Rightarrow 4\sin\theta - 3\cos\theta \equiv 5\sin(\theta - 36.87^0)$

$\Rightarrow f(x) = \frac{1}{2+5\sin(\theta-36.87^0)}$

The denominator has a range from -3 to 7 (Remember that $1 \div 0 = \infty$) Therefore, $f(x)$ has a range $f(x) \ge \frac{1}{7}$ and $f(x) \le -\frac{1}{3}$

(ii) The maximum point is found where $\sin(\theta - 36.87^0)$ is $+1$ that is ,

$\theta - 36.87^0 = 360^0$, $\theta = 396.87^0$. therefore the maximum point is $\left(396.87^0, -\frac{1}{3}\right)$

(iii) The minimum point is found where $\sin(\theta - 36.87^0)$. therefore the minimum is -1. That is $\theta - 36.87^0 = 180^0, \theta = 216.87^0$.

Therefore the minimum point is $\left(216.87^0, \frac{1}{7}\right)$

Exercise 5.4

1. Express each of the following in the form $R\cos(x + \alpha)$ with $0^0 \le x \le 360^0$

(a) $3\cos x - 4\sin x$

(b) $2\cos x - 5\sin x$

(c) $3\cos x - 2\sin x$

2. Express each of the following in the form $R\sin(x + \alpha)$ with $0^0 \le x \le 360^0$

(a) $7\sin x + 24\cos x$

(b) $3\sin x + \cos x$

(c) $5\sin x + 3\cos x$

3. Find the maximum value of each of the following expressions and the smallest positive value of θ that gives this maximum value.

(a) $\cos\theta + \sin\theta$

(b) $3\cos\theta + 4\sin\theta$

(c) $8\sin\theta + 15\cos\theta$

(d) $4\cos\theta - 3\sin\theta$

(e) $3\sqrt{2}\cos\theta + 7\sin\theta$

(f) $\sqrt{3}\sin\theta + \cos\theta$

(g) $\cos(\theta + 60^0) - \cos\theta$

4. Express $4\sin\theta - 3\cos\theta$ in the form $r\sin(\theta - \alpha)$. Hence find the maximum and minimum value of $\frac{7}{4\sin\theta - 3\cos\theta + 2}$. State the greatest and least values.
5. Express$\cos x + \sin x$ in the form $r\cos(x - \alpha)$. Hence find the smallest positive values of $x$for which $\frac{1}{(\cos x + \cos x)}$ has a minimum value.
6. Express $3\cos x - 4\sin x$ in the form $r\cos(x + \alpha)$. Hence express $4 + \frac{10}{3\cos x - 4\sin x}$in the form $4 + k\sec(x + \alpha)$.
7. Solve the following equations for $0^0 \le x \le 360^0$

(a) $8\cos x - 15\sin x = 5$

(b) $8\cos x - 12\sin x = 10$

(c) $2\cos x - \sin x = 1$

(d) $5\cos x + 2\sin x = 2$

(e) $3\sin x - 5\cos x = -4$

(f) $2\cos 2x - 4\sin x\cos x = \sqrt{6}$

8. Solve the following equations for values of θ from 0^0 to 360^0 inclusive.

(a) $\sqrt{3}\cos\theta + \sin\theta = 1$

(b) $5\sin\theta - 12\cos\theta = 6$

(c) $\sin\theta + \cos\theta = \frac{1}{2}$

(d) $\cos\theta - 7\sin\theta = 2$

(e) $2\sin\theta + 7\cos\theta = 4$

(f) $3\tan\theta - 2\sec\theta = 4$

(g) $4\cos\theta\sin\theta + 15\cos\theta = 10$

(h) $\cos\theta + \sin\theta = \sec\theta$

9. Prove that $\cos\theta - \sin\theta = \sqrt{2}\cos(\theta + 45^0) = -\sqrt{2}\sin(\theta - 45^0)$
10. Solve the following equations for $0^0 \le \theta \le 360^0$

(a) $\cos\theta + 7\sin\theta = 5$

(b) $3\cos\theta - 4\sin\theta = 5$

11. Find the greatest and least value of each of the following expressions, and state, correct to one decimal place, the smallest non negative value of θ for which each occurs.

(a) $12\sin\theta + 5\cos\theta$ $Ans(13, 67.4^0;\ -13, 247.4^0)$

(b) $2\cos\theta + \sin\theta$ $Ans(\ \sqrt{5}, 26.6^0;\ -\sqrt{5}, 206.6^0)$

(c) $7 + 3\sin\theta - 4\cos\theta$ $Ans(\ 12, 143.1^0; 2\ \ , 232.1^0)$

(d) $10 - 2\sin\theta + \cos\theta$ $Ans(\ 10 + \sqrt{5}, 296.6^0;\ \ 10 - \sqrt{5}, 116.6^0)$

(e) $\frac{1}{2 + \sin\theta + \cos\theta}$ $Ans(\ \frac{1}{2 - \sqrt{2}}, 225^0;\ \frac{1}{2 + \sqrt{2}}\ \ , 45^0)$

(f) $\frac{1}{7 - 2\cos\theta + \sqrt{5}\sin\theta}$ $Ans(\frac{1}{4}, 311.8^0;\ \frac{1}{10}, 131.8^0)$

(g) $\frac{3}{5\cos\theta - 12\sin\theta + 16}$ $Ans(\ 1, 112.6^0;\ \frac{3}{29}, 292.6^0)$

(h) $\frac{2}{7-4\sqrt{3}\cos\theta+\sin\theta}$ $Ans(\ \pm\infty, near\ 351.8^0)$

12. Solve the following equations for $0^0 \le \theta \le 360^0$, giving your answers correct to one decimal place.

(a) $\sin\theta + \sqrt{3}\cos\theta\ Ans(90^0, 330^0)$

(b) $\cot\theta - \sqrt{13}cosec\theta Ans(236.3^0, 326.3^0)$

(c) $24 = 10cosec\theta - 7\cot\theta\ Ans(7.3^0, 140.2^0)$

(d) $\frac{\sqrt{5}}{2}\sec\theta - \tan\theta = 2Ans(86.6^0, 326.6^0)$

(e) $\sqrt{2}\tan\theta - \sqrt{3}\sec\theta\ Ans(52.5^0, 82.5^0, 232.5^0, 262.5^0)$

13. Show that $1-\sqrt{2} \le 2cos^2\theta + \sin 2\theta \le 1+\sqrt{2}$ for all values of θ

14. (a) Given that $6cos^2\theta - 8\sin\theta\cos\theta \equiv A + R\cos(2\theta+\alpha)$, find the values of the constants A, R and α. $Ans(A = 3, R = 5, \alpha = \tan^{-1}\frac{4}{3})$

(b) Hence solve the equation $6cos^2\theta - 8\sin\theta\cos\theta = 5$, for $0^0 \le \theta \le 360^0$.

$Ans(6.6^0, 120.2^0, 186.6^0, 300.2^0)$

15. Solve the equation $\sqrt{3}\tan\theta - \sec\theta = 1$, giving all solutions in the interval $0^0 < \theta < 360^0$. $Ans(60^0, 180^0)$

16. The function f is defines for all real values of x by

$$f(x) = (\cos x - \sin x)(17\cos x - 7\sin x)$$

(a) By multiplying out the brackets, show that $f(x)$ may be expressed in the form $5\cos 2x - 12\sin 2x + k$ where k is a constant, and state the value of k.

(b) Given that $5\cos 2x - 12\sin 2x \equiv R\cos(2x+\alpha)$, where $R > 0$ and $0^0 < \alpha < 90^0$, state the value of R and find the value of α in degrees.

(c) Determine the greatest and least values of $\frac{39}{f(x)+14}$ and state a value of x at which the greatest value occurs.

5.5 t –Formulae.

- Equations of the form $a\cos x + b\sin x = c$ can also be solved using the substitution $t = \tan\frac{x}{2}$.
- From the double angle formulae

1. $\cos x = cos^2\frac{x}{2} - sin^2\frac{x}{2}$

$= \frac{cos^2\frac{x}{2} - sin^2\frac{x}{2}}{1}$

$= \frac{cos^2\frac{x}{2} - sin^2\frac{x}{2}}{cos^2\frac{x}{2} + sin^2\frac{x}{2}}$

$= \frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}$ dividing through by $cos^2\frac{x}{2}$

$= \frac{1-t^2}{1+t^2}$

2. $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1}$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{cos^2\frac{x}{2}+sin^2\frac{x}{2}}$$

$$= \frac{2\tan\frac{x}{2}}{1+tan^2\frac{x}{2}}$$ dividing through by $cos^2\frac{x}{2}$

$$= \frac{2t}{1+t^2}$$

3. $\tan x = \frac{\sin x}{\cos x}$

$$= \frac{\left(\frac{2t}{1+t^2}\right)}{\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \frac{2t}{1-t^2}$$

Note: If the cosine and sine are in terms of $2x$ then we use the substitution $t = \tan x$

Example 42

Solve the equation $5\cos x - 2\sin x = 2$ for $-180^0 \leq x \leq 180^0$ using the substitution $t = \tan\frac{x}{2}$

Solution

$\Rightarrow 5\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right) = 2$

$\Rightarrow 5 - 5t^2 - 4t = 2 + 2t^2$

$\Rightarrow 7t^2 + 4t - 3 = 0$

$\Rightarrow t = \frac{-4\pm\sqrt{4^2-4(7)(-3)}}{2(7)}$

$\Rightarrow t = -1 \quad or\ t = \frac{3}{7}$

Hence $\tan\frac{x}{2} = -1$ or $\tan\frac{x}{2} = 3/7$

$\Rightarrow \frac{x}{2} = -45^0, 135^0;\ \ x = -90^0, 270^0$ $\frac{x}{2} = -156.80^0, 23.20^0;\ \ x = -313.60^0, 46.40^0$

∴the solutions in the range $-180^0 \leq x \leq 180^0$ are $x = -90^0, 46.40^0$.

Qn. Solve the equation $5\cos 2x - 12\sin 2x = 3$ for $0^0 \leq x \leq 360^0$ using the substitution $t = \tan x$.

Exercise 5.5

1. Solve the following equations for $0^0 \leq x \leq 360^0$

(a) $2\cos x + 3\sin x - 2 = 0 Ans(0^0, 112.6^0, 360^0)$

(b) $7\cos x + \sin x - 5 = 0 Ans(53.1^0, 323.1^0)$

(c) $3\cos x - 4\sin x + 1 = 0 Ans(48.4^0, 205.3^0)$

(d) $3\cos x + 4\sin x = 2 Ans(119.6^0, 346.7^0)$

2. Use the t-formulae to solve the equations

(a) $7\cos x + 6\sin x = 2$ for $0^0 \leq x \leq 360^0 Ans(118.1^0, 323.1^0)$

(b) $9\cos x - 8\sin x = 12$ for $-360^0 \leq x \leq 360^0$ $Ans(-46.4^0, -36.9^0, 313.6^0, 323.1^0)$

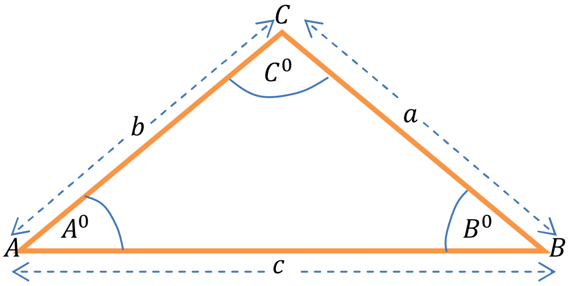
3. By expressing $\cos 2\theta$ and $\sin 2\theta$ in terms of $\tan\theta$, solve the following equations for $0^0 \leq \theta \leq 360^0$

(a) $\cos 2\theta - 2\sin 2\theta = 2$ $Ans(135^0, 161.6^0, 315^0, 341.6^0)$

(b) $5\cos 2\theta - 2\sin 2\theta = 2$ $Ans(23.2^0, 135^0, 203.2^0, 315^0)$

5.6 Solution of Triangles

- In the notation (Euler's notation) the vertices are always labelled with capital letters, say A, B and C, and he same symbols are used to represent the sizes of the angles at these vertices. The corresponding lower case letters, a, b and c are then used to represent the lengths of the sides opposite the vertices, i.e. the letter a is used to represent the length of the side BC as seen in the following diagram.



- The units of angles are degrees. The sub-unit is minute, which is $1/60^{th}$ of a degree, and the standard symbol for it is a small dash. So $35^0 12' = 35\frac{12^0}{60}$; in decimals this becomes 35.2^0.

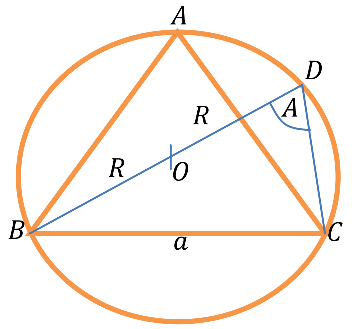
The sine rule:

In the triangle ABC; $\frac{\boldsymbol{a}}{\sin \boldsymbol{A}} = \frac{\boldsymbol{b}}{\sin \boldsymbol{B}} = \frac{\boldsymbol{c}}{\sin \boldsymbol{C}} = \boldsymbol{2R}$is the sine rule.

Proof

Let ABC be any triangle inscribed in a circle of radius R.

fig (a)



Let O be the centre of the circle.

Joining BO and produced, it meets the circumference in D. Join CD

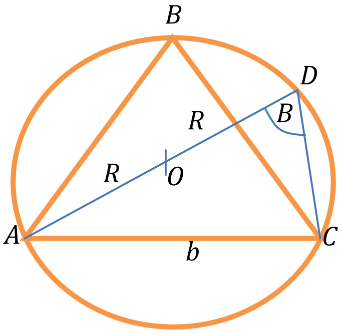
$\angle BCD = 90^0$ (angle in the semi-circle)

$\angle BDC = \angle A$ (angles in the same segment)

$\frac{BC}{BD} = \sin BDC = \sin A$ or $\frac{a}{2R} = \sin A \therefore \boldsymbol{\frac{a}{\sin A} = 2R}$............(i)

Interchanging A with B

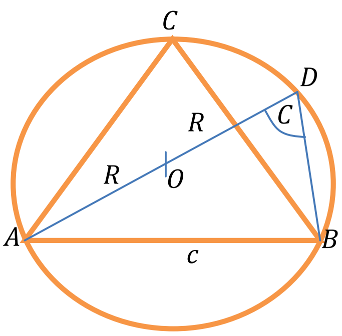
fig (b)



$\frac{AC}{AD} = \sin ADC = \sin B$ or $\frac{b}{2R} = \sin B \therefore \boldsymbol{\frac{b}{\sin B} = 2R}$............(ii)

Interchanging B with C

fig (c)



$\frac{AB}{AD} = \sin ADB = \sin C$ or $\frac{c}{2R} = \sin C \therefore \boldsymbol{\frac{c}{\sin C} = 2R}$............(iii)

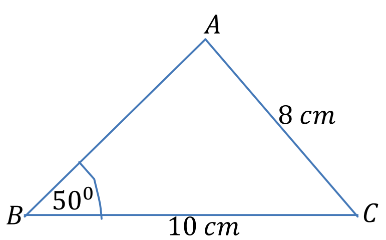
Equating equations (i), (ii) and (iii)

$\therefore \boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R}$ ....■

Example 43

In the triangle ABC, $a = 10\ cm, b = 8\ cm, B = 50^0$. Find A, C and c

Solution



Using the sine rule;

$\frac{10}{\sin A} = \frac{8}{\sin 50^0} = \frac{c}{\sin C}$

In this case

$\frac{10}{\sin A} = \frac{8}{\sin 50^0}$

$A = \sin^{-1}\left(\frac{10\sin 50^0}{8}\right) = 73.25^0 or\ 106.75^0$

$\Longrightarrow A + B + C = 180^0$

$\Longrightarrow 73.25^0 + 50^0 + C = 180^0 \therefore C = 56.75^0$

Using $\frac{8}{\sin 50^0} = \frac{c}{\sin 56.75^0}$

$\Longrightarrow c = \frac{8\sin 56.75^0}{\sin 50^0} = 8.7336\ cm$

$\therefore A = 73.25^0, C = 56.75^0 and\ c = 8.7336\ cm$

Example 44

Prove that in any triangle ABC, $\frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$.

Solution

Using the sine rule; $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\Longrightarrow a = 2R\sin A, b = 2R\sin B, c = 2R\sin C$

From the LHS

$\Longrightarrow \frac{a^2-b^2}{c^2} = \frac{(2R\sin A)^2-(2R\sin B)^2}{(2R\sin C)^2}$

$= \frac{sin^2A-sin^2B}{sin^2C}$

Since $C = 180^0 - A - B$; $\sin C = \sin(180^0 - A - B) = \sin(A + B)$

$\frac{a^2-b^2}{c^2} = \frac{(\sin A+\sin B)(\sin A-\sin B)}{sin^2(A+B)}$

$= \frac{2\left(\sin\frac{A+B}{2}\cos\frac{A-B}{2}\right)2\left(\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right)}{sin^2(A+B)}$

$= \frac{2\left(\sin\frac{A+B}{2}\cos\frac{A+B}{2}\right)2\left(\sin\frac{A-B}{2}\cos\frac{A-B}{2}\right)}{sin^2(A+B)}$

$= \frac{\sin(A+B)\sin(A-B)}{sin^2(A+B)}$

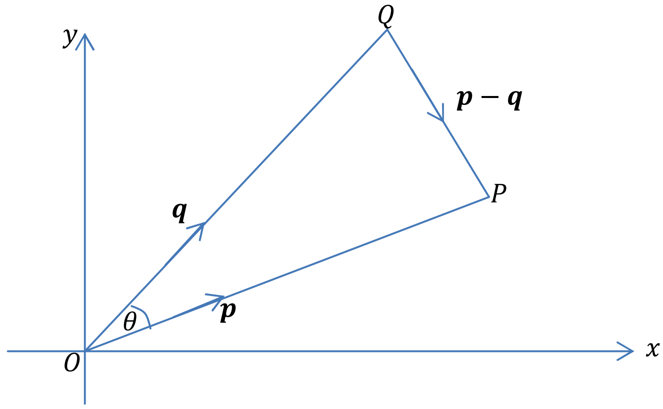
$= \frac{\sin(A-B)}{\sin(A+B)}$

$= RHS$

$\therefore \frac{a^2-b^2}{c^2} = \frac{\sin(A-B)}{\sin(A+B)}$ ……………..■

The Cosine rule:

The cosine rule has very many proofs. We can use the one which includes the idea of the scalar product.



In the triangle OPQ, the angle POQ is equal to θ and $\overrightarrow{QP} = \boldsymbol{p} - \boldsymbol{q}$.

Consider the scalar product $\overrightarrow{QP}.\overrightarrow{QP}$.

$\Rightarrow \overrightarrow{QP}.\overrightarrow{QP} = (\boldsymbol{p}-\boldsymbol{q}).(\boldsymbol{p}-\boldsymbol{q})$

$= \boldsymbol{p}.\boldsymbol{p} + \boldsymbol{q}.\boldsymbol{q} - 2\boldsymbol{p}.\boldsymbol{q}$

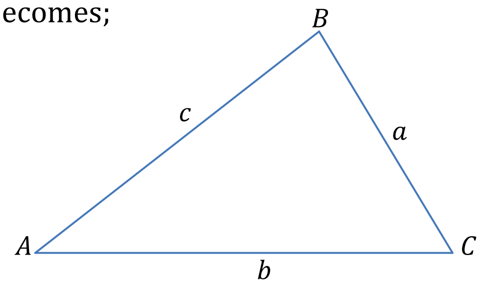
$= p^2 + q^2 - 2pq\cos\theta$; $\boldsymbol{p}.\boldsymbol{q} = |\boldsymbol{p}|.|\boldsymbol{q}|\cos\theta$

But $\overrightarrow{QP}.\overrightarrow{QP}$ is equal to QP^2,

$\therefore QP^2 = p^2 + q^2 - 2pq\cos\theta$

So, if we are given the values of p and q, and the size of the included angle θ, we can calculate the length of QP.

Using the Euler's notation the triangle POQ is re-lettered ABC and the cosine rule becomes;



$a^2 = b^2 + c^2 - 2bc\cos A$

The letters a, b and c are permuted to give the following alternative forms:

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Note: The cosine rule works well when given the length of two sides and an included angle.

Example 45

In the triangle PQR, $p = 14.3\ r = 17.5\ and\ Q = 25^0$
$36'$. Calculate the length of side PR
Solution:
Using $q^2 = r^2 + p^2 - 2rp\cos Q$
$\Longrightarrow q^2 = 17.5^2 + 14.3^2 - 2 \times 17.5 \times 14.3\cos 25.6^0$
$\therefore q = 7.7054$

Example 46

In the triangle XYZ, $x = 4\ cm, y = 6\ cm\ and\ z = 3\ cm$. Calculate the size of angle Y.
Solution
Using $y^2 = z^2 + x^2 - 2zx\cos Y$
Using $6^2 = 3^2 + 4^2 - 2(3)(4)\cos Y$
$\therefore Y = 117.28^0$.

The tangent formulae:
We now derive another set of formulae which are useful in the numerical solution of triangles. Starting from the sine formula, $b = 2R\sin B$, $c = 2R\sin C$, and after division of numerator and denominator by "R, we have
$\frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C}$
$= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$
$= \cot\frac{B+C}{2}\tan\frac{B-C}{2}$
Hence $\tan\frac{1}{2}(B - C) = \left(\frac{b-c}{b+c}\right)\tan\frac{1}{2}(B + C)$
Since $\frac{1}{2}(B + C) = 90^0 - \frac{1}{2}A$, this can be written as $\tan\frac{1}{2}(B - C) = \left(\frac{b-c}{b+c}\right)\cot\frac{1}{2}A$
The two corresponding formulae are
$\tan\frac{1}{2}(C - A) = \left(\frac{c-a}{c+a}\right)\cot\frac{1}{2}B$
$\tan\frac{1}{2}(A - B) = \left(\frac{a-b}{a+b}\right)\cot\frac{1}{2}C$
And these can be similarly derived.

Example 47

Prove that in any triangle ABC, $\sin\frac{1}{2}(B-C) = \left(\frac{b-c}{a}\right)\cos\frac{1}{2}A$

Solution

From the sine rule; $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$

From $\left(\frac{b-c}{a}\right) = \frac{2R\sin B - 2R\sin C}{2R\sin A}$

$= \frac{\sin B - \sin C}{\sin A}$

$= \frac{\sin B - \sin C}{\sin(B+C)}$

Since $A = 180^0 - B - C$

$\left(\frac{b-c}{a}\right) = \frac{2\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B-C)}{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B+C)}$

$= \frac{\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}(B+C)}$

$= \frac{\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}A}$ since $\frac{1}{2}(B-C) = 90^0 - \frac{1}{2}A$

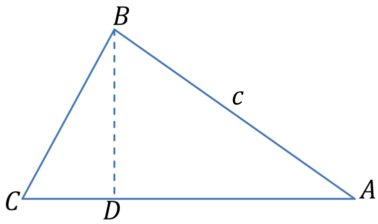
We multiply through by $\cos\frac{1}{2}A$

$\therefore \sin\frac{1}{2}(B-C) = \left(\frac{b-c}{a}\right)\cos\frac{1}{2}A$ ……….■

The area of the triangle:

(i) *When given two sides and the included angle.*

Let Δ denote the area of the triangle ABC and let BD be the perpendicular from B on AC.



Then since $BD = c\sin A$

$\Longrightarrow \Delta = \frac{1}{2}CA.BD = \frac{1}{2}bc\sin A$

In the same way we could show that $\Delta = \frac{1}{2}ac\sin B$, $\Delta = \frac{1}{2}ab\sin C$.

From $\Delta = \frac{1}{2}bc\sin A$, expressing this in the form $\frac{a}{\sin A} = \frac{\frac{1}{2}abc}{\Delta}$, we could rewrite the sine formula as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = \frac{abc}{2\Delta}$. We can use this to find the **radius** of the circumscribing circle.

(ii) *When given the three sides of a triangle.*

To find the area of the triangle in term of the sides we consider the following

$\frac{1}{2}bc\sin A = \Delta$

$\Longrightarrow 2bc\sin A = 4\Delta$ ……………..(1)

From the cosine rule; $2bc\cos A = b^2 + c^2 - a^2$……………(2)

Squaring and adding equations (1) and (2)

$\Longrightarrow (2bc\sin A)^2 + (2bc\cos A)^2 = (4\Delta)^2 + (b^2 + c^2 - a^2)^2$

$\Longrightarrow 4b^2c^2 = 16\Delta^2 + (b^2 + c^2 - a^2)^2$

$\Longrightarrow \Delta^2 = \frac{1}{16}\{4b^2c^2 - (b^2 + c^2 - a^2)^2\}$using the difference of two squares;

$= \frac{1}{16}\{(2bc + (b^2 + c^2 - a^2))(2bc - (b^2 + c^2 - a^2))\}$

$= \frac{1}{16}\{((b+c)^2 - a^2)(a^2 - (b-c)^2)\}$

$= \frac{1}{16}\{(b+c+a)(b+c-a)(a-b+c)(a+b-c)\}$

If we write $2s = a + b + c$; where s is half the perimeter of the triangle

$\Longrightarrow \Delta^2 = \frac{1}{16}.2s.(2s-2a)(2s-2b)(2s-2c)$

$\therefore \Delta = \sqrt{\boldsymbol{s(s-a)(s-b)(s-c)}}$this is also called the **Hero's rule.**

Example 48

Given that the sides of a triangle are of length $a = 3.57\ cm,\ b = 2.61\ cm, c = 4.72\ cm$. Find its area and the radius of its circumference.

Solution

From $2s = a + b + c = 3.57 + 2.61 + 4.72$

$= 10.90$

$s = 5.45$

Hence from the formula $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Longrightarrow \Delta = \sqrt{[5.45(5.45 - 3.57)(5.45 - 2.61)(5.45 - 4.72)]}$

$= 4.6089\ cm^2$

From $\Delta = \frac{abc}{4R}$

$\Longrightarrow R = \frac{abc}{4\Delta}$

$\Longrightarrow R = \frac{3.57\times 2.61\times 4.72}{4\times 4.6089}$

$\therefore R = 2.3856\ cm$

Formulae for the angles of a triangle in terms of the sides

By rearranging the cosine formula $a^2 = b^2 + c^2 - 2bc\cos A$ in the form $\cos A = (b^2 + c^2 - a^2)/2bc$.

We can express the angle A in terms of the sides of a triangle.

1. $\sin\frac{1}{2}A = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}$

2. $\cos\frac{1}{2}A = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$
3. $\tan\frac{1}{2}A = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}$

Example 49

Prove that in a triangle ABC, $\frac{1}{2}cos^2\frac{1}{2}A + \frac{1}{2}cos^2\frac{1}{2}B + \frac{1}{2}cos^2\frac{1}{2}C = \frac{(a+b+c)^2}{4abc}$

Solution

From the LHS

$$\Rightarrow \frac{1}{2}cos^2\frac{1}{2}A + \frac{1}{2}cos^2\frac{1}{2}B + \frac{1}{2}cos^2\frac{1}{2}C = \frac{s(s-a)+s(s-b)+s(s-c)}{abc}$$
$$= \frac{3s^2 - s(a+b+c)}{abc}$$
$$= \frac{3s^2 - 2s^2}{abc}$$
$$= \frac{s^2}{abc}$$
$$= \frac{(a+b+c)^2}{4abc}$$
$$= RHS$$

$\therefore \frac{1}{2}cos^2\frac{1}{2}A + \frac{1}{2}cos^2\frac{1}{2}B + \frac{1}{2}cos^2\frac{1}{2}C = \frac{(a+b+c)^2}{4abc}$ ………………■

Exercise 5.6

Sine rule:

1. Solve the triangle in which $c = 26.83m, A = 80^0 30', B = 40^0 12'$.
   $Ans(C = 59^0 18', a = 3.77\ m, b = 20.14\ m$
2. Is there a triangle in which $b = 5, c = 7\ and\ B = 48^0 35'$? If so solve the triangle.
3. Solve the triangle in which $b = 5.6, c = 7.0\ and\ B = 53^0 8'$
   $Ans(A = 36^0 52', C = 90^0, a = 4.2)$
4. Solve the triangle in which $b = 24.93\ m, c = 12.10\ m, B = 122^0 51^0$
   $Ans(A = 33^0 5', C = 24^0 4', a = 16.20\ m)$
5. Use the sine formula to prove that, in triangle ABC,

(a) $\cos\frac{1}{2}(B - C) = \frac{b+c}{a}\sin\frac{1}{2}A$

(b) $\frac{a+b-c}{a-b+c} = \tan\frac{1}{2}B\cos\frac{1}{2}C$

(c) $\frac{a+b+c}{a+b-c} = \cot\frac{1}{2}A\cot\frac{1}{2}B$

Cosine rule:

6. Solve the triangle in which $b = 10.67\ m, c = 21.7\ m, A = 44^0 46'$
   $Ans(a = 16.0\ m, B = 28^0 1', C = 107^0 13')$
7. Solve the triangle in which $a = 16\ m, b = 10.67\ m, c = 21.7\ m$
8. Calculate the remaining angles of a triangle in which two sides are $13.46\ m, 54.31\ m$ and the included angle is $67^0 24'$.

Tangent rule

9. Solve the triangle in which $b = 10.67\ m, c = 21.7\ m, A = 44^0 46'$

$$Ans(B = 28^0\ C = 107^0 14', A = 16.0\ m)$$

Area of triangle

10. Find the area of a triangle having sides of length $322.2\ m, 644.7\ m\ and\ 432.1\ m$.
11. Find the area of a triangle whose sides are 10.4, 12.8and 17.6 m.
12. Find the area of a triangle with sides 4.2 cm and 3.5 cm and the included angle of 51.2^0.

SENIOR FIVE TERM TWO
CHAPTER 6
VECTORS

6.1Vector algebra [Vectors in Two and Three Dimensions]

Definition of a vector

- A **vector** is a physical quantity with both magnitude and direction e.g. force, acceleration, velocity, displacement etc.
- A scalar quantity is a physical quantity with only magnitude. E.g. mass weight, height, time etc.

Vector notation

- A vector AB is represented in the following ways;

  $\underset{\sim}{\overrightarrow{AB}}$, ***AB***but this is done when typing examinations
- Thus $\overrightarrow{AB}$ indicates both length and direction of **AB.**

Displacement

- Consider two points A and B with coordinates $(x_1, y_1)$and (x_2, y_2). The displacement vector $\overrightarrow{\mathrm{AB}}$ is given by

  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

  $= \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$

Example 1

Write down the displacement vector $\overrightarrow{AB}$ where $A(3, 5)$ and $B(5, 9)$

Solution

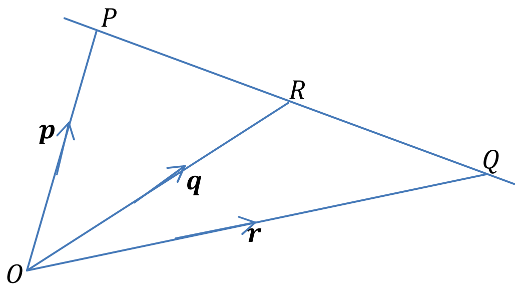
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \begin{pmatrix} 5 \\ 9 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Position vectors

- The position vectors of the points A, B,C, ... are given by $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, ... with respect to the origin.
- If a point P has coordinates (x, y), its position vector is given by $\begin{pmatrix} x \\ y \end{pmatrix}$. It is called the *column vector* of P.
- Consider the points P, R, Q in a straight line



- In the figure above R is the point on the line PQ, such that $\overrightarrow{PR} = t\overrightarrow{PQ}$. The position vector of R can be obtained as follows;

  From $\overrightarrow{PR} = t\overrightarrow{PQ}$

  $\Rightarrow \overrightarrow{OR} - \overrightarrow{OP} = t\left(\overrightarrow{OQ} - \overrightarrow{OP}\right)$

  $\Rightarrow \boldsymbol{r} - \boldsymbol{p} = t(\boldsymbol{q} - \boldsymbol{p})$

  $\Rightarrow \boldsymbol{r} = \boldsymbol{p} + t\boldsymbol{q} - t\boldsymbol{p}$

  $\Rightarrow \boldsymbol{r} = (1 - t)\boldsymbol{p} + t\boldsymbol{q}$is the position vector of R.

Magnitude of a vector

- The magnitude of a vector is the same as length or modulus.
- Consider a vector $\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ its magnitude is denoted as $|\boldsymbol{a}|$ where

  $|\boldsymbol{a}| = \sqrt{x^2 + y^2}$
- This can be extended to three dimensional space where $a = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and its magnitude is given by $|\boldsymbol{a}| = \sqrt{x^2 + y^2 + z^2}$.

Example 2

Find the magnitude of the following vectors;

(a) $\boldsymbol{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ (b) $\boldsymbol{b} = \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}$

Solution

(a) $|\boldsymbol{a}| = \sqrt{4^2 + 2^2} = \sqrt{20} = 4.4721\ units$

(b) $|\boldsymbol{b}| = \sqrt{(2)^2 + (-4)^2 + (5)^2} = \sqrt{45} = 6.7082\ units$

Unit vectors

- Is a vector of magnitude one unit in a given direction.

- If we denote a horizontal unit by $\boldsymbol{i}$and a vertical unit by $\boldsymbol{j}$then $\overrightarrow{OA} = x\boldsymbol{i} + y\boldsymbol{j}$.
- A unit vector in the direction of some vector $\boldsymbol{a}$is usually written as $\hat{\boldsymbol{a}}$where $\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$.

Example 3

Find the unit vector in the same direction as the vector $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$

Solution

$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$

$= \frac{2i-j}{\sqrt{(2)^2+(-1)^2}} = \frac{2}{\sqrt{5}}\boldsymbol{i} - \frac{1}{\sqrt{5}}\boldsymbol{j}$

Example 4

Find the vector of magnitude 20 units and is parallel to the vector$\boldsymbol{b} = 3\boldsymbol{i} - 4\boldsymbol{j}$.

Solution

Let the required vector be $\boldsymbol{v}$

$\Rightarrow \boldsymbol{v} = 20\hat{\boldsymbol{b}}$

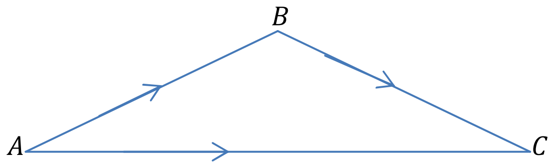
$= 20\left[\frac{3\boldsymbol{i}-4\boldsymbol{j}}{\sqrt{(3)^2+(-4)^2}}\right]$

$= 20\left[\frac{3\boldsymbol{i}-4\boldsymbol{j}}{5}\right]$

$= 12\boldsymbol{i} - 16\boldsymbol{j}$

Addition and subtraction of vectors

- The diagram below shows the path that could be taken to travel from A to C.



- $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
- The vector $\overrightarrow{AC}$ is called the resultant of vectors $\overrightarrow{AB}$and $\overrightarrow{BC}$.
- Since vectors may also be written as single letters in bold type, if we let $u = \overrightarrow{AB}, v = \overrightarrow{BC}$ and $w = \overrightarrow{AC}$, then we have $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$
- Since $\overrightarrow{AB} = \boldsymbol{u}$, we have $\overrightarrow{BA} = -\boldsymbol{u}$. In other words, the vector $\overrightarrow{AB}$ has the same magnitude as $\overrightarrow{BA}$ but is in the opposite direction.

Example 5

Given that $\overrightarrow{AB} = 3\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k}$and $\overrightarrow{BC} = -\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$. Find $\overrightarrow{AC}$.

Solution

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= (3\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k}) + (-\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$

$= (2\boldsymbol{i} + 9\boldsymbol{j} - 5\boldsymbol{k})$

Example 6

Given that $\overrightarrow{BC} = 7\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$and $\overrightarrow{AC} = \boldsymbol{i} - 6\boldsymbol{k}$, find $\overrightarrow{BA}$

Solution

$\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA}$

$= \overrightarrow{BC} - \overrightarrow{AC}$

$= (7\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) - (\boldsymbol{i} - 6\boldsymbol{k})$

$= 6\boldsymbol{i} - 2\boldsymbol{j} + 7\boldsymbol{k}$

Example 7

Two vectors are given by $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$and $\boldsymbol{b} = -\boldsymbol{i} + 3j + 4k$, find;

(a) $\boldsymbol{a} + \boldsymbol{b}$(b) $\boldsymbol{a} - \boldsymbol{b}$

Solution

(a) $\boldsymbol{a} + \boldsymbol{b} = (2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) + (-\boldsymbol{i} + 3j + 4k)$

$= \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

(b) $\boldsymbol{a} - \boldsymbol{b} = (2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}) - (-\boldsymbol{i} + 3j + 4k)$

$= 3\boldsymbol{i} - 4\boldsymbol{j} - 5\boldsymbol{k}$

Multiplication by a scalar

- Consider a vector $a = \begin{pmatrix} x \\ y \end{pmatrix}$, and a scalar k then,

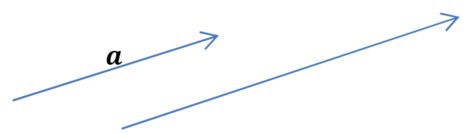
  $\Rightarrow k\boldsymbol{a} = k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$

- For example if $\boldsymbol{a} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$,

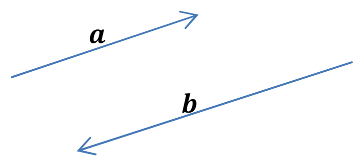
  $\Rightarrow 5\boldsymbol{a} = 5\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -15 \\ 20 \end{pmatrix}$

Equal vectors

- Two vectors **a** and **b** are said to be parallel if one is a scalar multiple of the other i.e. if$\boldsymbol{a} = \lambda\boldsymbol{b}$.
- If λ is positive, the vectors are parallel and in the same direction: i.e. they are ***like*** parallel vectors



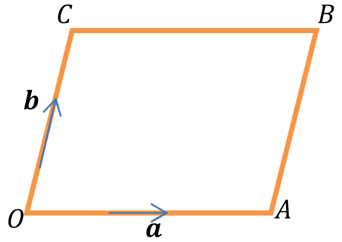
- If λ is negative, the vectors are parallel and in opposite directions: i.e. they are ***unlike*** parallel vectors.



Example 8

In the parallelogram OABC, $\overrightarrow{OA} = \boldsymbol{a}$and $\overrightarrow{OC} = \boldsymbol{c}$. The point D lies on AB and is such that AD:DB=1:2. Express the following vectors in terms of $\boldsymbol{a}$ and$\boldsymbol{c}$

(a) $\overrightarrow{CB}$(b) $\overrightarrow{BC}$ (c)$\overrightarrow{AB}$ (d) $\overrightarrow{AD}$ (e) $\overrightarrow{OD}$ (f) $\overrightarrow{DC}$



Solution

(a) $\overrightarrow{CB}$ is the same length as $\overrightarrow{OA}$ and is in same direction

$\therefore \overrightarrow{CB} = \overrightarrow{OA} = \boldsymbol{a}$

(b) $\overrightarrow{BC}$ is the same length as $\overrightarrow{OA}$ and is in opposite direction

$\therefore \overrightarrow{BC} = -\overrightarrow{CB} = -\boldsymbol{a}$

(c) $\overrightarrow{AB}$ is the same length as $\overrightarrow{OC}$ and is in same direction

$\therefore \overrightarrow{AB} = \overrightarrow{OC} = \boldsymbol{c}$

(d) D is one third of the way along AB,

$\overrightarrow{AD} = \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}\boldsymbol{c}$

(e) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$= \boldsymbol{a} + \frac{1}{3}\boldsymbol{c}$

(f) $\overrightarrow{DC} = \overrightarrow{DB} + \overrightarrow{BC}$

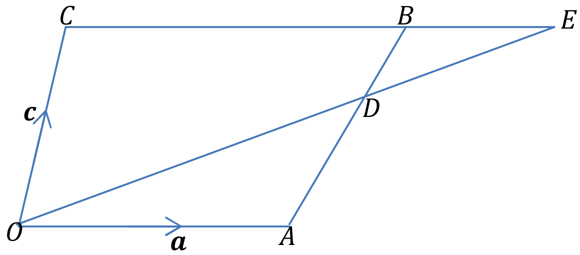
$= \frac{2}{3}\boldsymbol{c} + (-\boldsymbol{a}) = \frac{2}{3}\boldsymbol{c} - \boldsymbol{a}$

Non-parallel vectors

- For two non-parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ if $\lambda\boldsymbol{a} + \mu\boldsymbol{b} = \alpha\boldsymbol{a} + \beta\boldsymbol{b}$then $\lambda = \alpha$ and $\mu = \beta$ i.e. we equate the coefficients of **a** and **b** .

Example 9

The diagram show a parallelogram OABC with $\overrightarrow{OA} = \boldsymbol{a}$and $\overrightarrow{OC} = \boldsymbol{c}$. D is a point on AB such that AD:DB=2:1. OD produced meets CB produced at E. $\overrightarrow{DE} = h\overrightarrow{OD}$and $\overrightarrow{BE} = k\overrightarrow{CB}$.



Find;

(a) $\overrightarrow{BE}$ in terms of $\boldsymbol{a}$ and k.

(b) $\overrightarrow{DE}$ in terms of h, **a** and **c**.

(c) the values of h and k

Solution

(a) $\overrightarrow{BE} = k\overrightarrow{CB} = k\boldsymbol{a}$

(b) $\overrightarrow{DE} = h\overrightarrow{OD} = h\left(\boldsymbol{a} + \frac{2}{3}\boldsymbol{c}\right)$

(c) To determine h and k we need a vector equation containing h and k.

$\Rightarrow \overrightarrow{BE} = \overrightarrow{BD} + \overrightarrow{DE}$

$\Rightarrow k\boldsymbol{a} = -\frac{1}{3}\boldsymbol{c} + h\left(\boldsymbol{a} + \frac{2}{3}\boldsymbol{c}\right)$

$\Rightarrow k\boldsymbol{a} = -\frac{1}{3}\boldsymbol{c} + h\boldsymbol{a} + \frac{2}{3}h\boldsymbol{c}$

Equating coefficients of;

$\boldsymbol{a}; k = h$

$\boldsymbol{c}; 0 = -\frac{1}{3} + \frac{2}{3}h$

$\therefore h = 1/2$ and $k = 1/2$

Example 10

Show that the points A, B and C with position vectors $3\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$, $8\boldsymbol{i} + 7\boldsymbol{j} + 4\boldsymbol{k}$ and $11\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$ respectively are vertices of a triangle.

Solution

$\overrightarrow{AB} = (8\boldsymbol{i} + 7\boldsymbol{j} + 4\boldsymbol{k}) - (3\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$

$= (5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$

$\overrightarrow{BC} = (11\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}) - (8\boldsymbol{i} + 7\boldsymbol{j} + 4\boldsymbol{k})$

$= (3\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k})$

$\overrightarrow{AC} = (11\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}) - (3\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k})$

$= (8\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k})$

$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = (5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}) + (3\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k})$

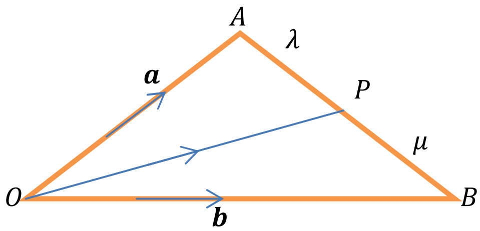
$= (8\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k})$

$= \overrightarrow{AC}$

$\therefore$ Since $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ and $\frac{AB}{BC} \neq \frac{BC}{AC} \neq a\ scalar$, then A, B and C are vertices of a triangle.

Ratio theorem

- Consider a triangle OAB and P divides the line AB in the ration $\lambda{:}\,\mu$ then the position vector of P can be expressed as shown below.



$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \boldsymbol{a} + \frac{\lambda}{\lambda+\mu}\overrightarrow{AB}$

$= \boldsymbol{a} + \frac{\lambda}{\lambda+\mu}(\boldsymbol{b} - \boldsymbol{a})$

$= \frac{(\lambda+\mu)\boldsymbol{a}+\lambda(\boldsymbol{b}-\boldsymbol{a})}{\lambda+\mu}$

$= \frac{\lambda\boldsymbol{a}+\mu\boldsymbol{a}+\lambda\boldsymbol{b}-\lambda\boldsymbol{a}}{\lambda+\mu}$

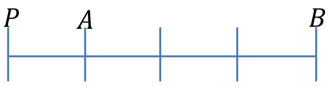
$= \frac{\mu\boldsymbol{a}+\lambda\boldsymbol{b}}{\lambda+\mu}$ is the position vector of P (ratio theorem)

- This theorem can be used to obtain the position vector of points that Divide lines

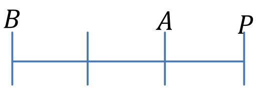
Division of a line by a point

- The point P can divide the line AB either internally or externally.
- If for example P divides AB internally in the ratio 1:3

- In this case $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are in the same direction i.e. $AP: PB = 1: 3$
- Point P dividing AB externally in the ratio 1:3 (By writing this as 1:-3 or -1:3 we do not state that the division is external)



(-1:3)



(1:-3)

- In this case the vector $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are in opposite direction. Hence the use of the minus sign in the ratio.

Example 11

The point K has position vector $3\boldsymbol{i}+2\boldsymbol{j}$and point L has the position vector$\boldsymbol{i}+3\boldsymbol{j}$, find the position vector of the point P which divides KL in the ratio.

(a) $4:3$

(b) $4:-3$

Solution

(a) Using the ratio theorem; $4:3 \Leftrightarrow \lambda:\mu$

$\Rightarrow \overrightarrow{OP} = \frac{\mu\boldsymbol{k}+\lambda\boldsymbol{l}}{\lambda+\mu}$

$= \frac{3(3\boldsymbol{i}+2\boldsymbol{j})+4(\boldsymbol{i}+3\boldsymbol{j})}{4+3}$

$= \frac{9\boldsymbol{i}+6\boldsymbol{j}+4\boldsymbol{i}+12\boldsymbol{j}}{7}$

$= \frac{13}{7}\boldsymbol{i}+\frac{18}{7}\boldsymbol{j}$

(b) Using the ratio theorem; $4:-3 \Leftrightarrow \lambda:\mu$

$\Rightarrow \overrightarrow{OP} = \frac{\mu\boldsymbol{k}+\lambda\boldsymbol{l}}{\lambda+\mu}$

$= \frac{-3(3\boldsymbol{i}+2\boldsymbol{j})+4(\boldsymbol{i}+3\boldsymbol{j})}{4-3}$

$= \frac{-9\boldsymbol{i}-6\boldsymbol{j}+4\boldsymbol{i}+12\boldsymbol{j}}{1}$

$= -5\boldsymbol{i}+6\boldsymbol{j}$

Example 12

If the position vectors of points A and B are $\begin{pmatrix}2\\4\\6\end{pmatrix}$ and $\begin{pmatrix}-3\\2\\8\end{pmatrix}$ respectively, find the position vector of the point R which divides AB externally in the ratio 5: 3

Solution

Using $\overrightarrow{OR} = \frac{\mu\boldsymbol{a}+\lambda\boldsymbol{b}}{\lambda+\mu}$; $-5:3 \Leftrightarrow \lambda:\mu$

$= \frac{3\begin{pmatrix}2\\4\\6\end{pmatrix}-5\begin{pmatrix}-3\\2\\8\end{pmatrix}}{-5+3}$

$= \frac{\begin{pmatrix}6\\12\\18\end{pmatrix}-\begin{pmatrix}-15\\10\\40\end{pmatrix}}{-2}$

$= -\frac{1}{2}\begin{pmatrix}21\\2\\-22\end{pmatrix}$

$= \begin{pmatrix}-10.5\\-1\\11\end{pmatrix}$

Other base vectors

- We can express a give vector in terms of other vectors a part from$\boldsymbol{i}, \boldsymbol{j}\ and\ \boldsymbol{k}$.
- If $\boldsymbol{a}, \boldsymbol{b}\ and\ \boldsymbol{c}$are coplanar vectors,

  $\Rightarrow \boldsymbol{c} = \lambda_1\boldsymbol{a}+\mu_1\boldsymbol{b}, \boldsymbol{b} = \lambda_2\boldsymbol{a}+\mu_2\boldsymbol{c}, \boldsymbol{a} = \lambda_3\boldsymbol{b}+\mu_3\boldsymbol{c}$where$\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \mu_3$ are suitable scalars.

- The word coplanar means to lie in the same plane.

Example 13

With $a = \binom{3}{5}$ and $b = \binom{2}{-1}$ as base vector express $c = \binom{5}{17}$ in the form $\lambda\boldsymbol{a} + \mu\boldsymbol{b}$

Solution

$\boldsymbol{c} = \lambda\boldsymbol{a} + \mu\boldsymbol{b}$

$\Rightarrow \binom{5}{17} = \lambda\binom{3}{5} + \mu\binom{2}{-1}$

$\Rightarrow 3\lambda + 2\mu = 5$ ……………….(1)

$5\lambda - \mu = 17$…………………….(2)

$2 \times eqn(2) + eqn(1)$

$\Rightarrow 13\lambda = 39;\ \lambda = 3, \mu = -2$

$\therefore \boldsymbol{c} = 3\boldsymbol{a} - 2\boldsymbol{b}$

Example 14

Show that the vectors $a = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, b = \begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$ $and\ c = \begin{pmatrix} 1 \\ -10 \\ 7 \end{pmatrix}$ are coplanar.

Solution

Suppose $\boldsymbol{c} = \lambda\boldsymbol{a} + \mu\boldsymbol{b}$

$\Rightarrow \begin{pmatrix} 1 \\ -10 \\ 7 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$

Then $1 = \lambda + 6\mu, -10 = -3\lambda - 4\mu\ and\ 7 = 2\lambda + 2\mu.$

Solving the first two equations simultaneously gives $\lambda = 4$ and $\mu = -\frac{1}{2}$

These values are compatible with $7 = 2\lambda + 2\mu$

Thus we can write $\boldsymbol{c} = 4\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$**,** hence **a**, **b** and **c** are coplanar.

Collinear vectors

- To be collinear means to lie on the same straight line i.e. if we take three points A, B and C
- Then A, B and C are said to be collinear points if $\frac{AB}{AC} = k$ where k is a scalar and the two have a common point A.

Example 15

The position vectors of the points A, B and C are $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $6\boldsymbol{i} + 11\boldsymbol{j} - 7\boldsymbol{k}$respectively. Show that A, B and C are collinear.

Solution

$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} 6 \\ 11 \\ -7 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

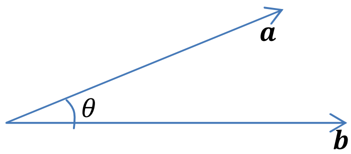
$$= \begin{pmatrix} 3 \\ 9 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$\therefore \overrightarrow{BC} = 3\overrightarrow{AB}$$

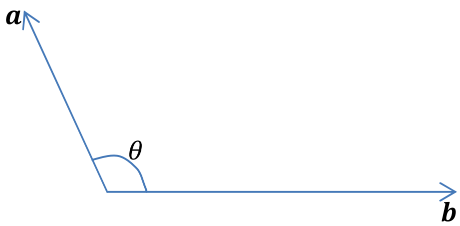
It is clear that since $\overrightarrow{BC}$ is a multiple of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AB}$ are parallel. But $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have a common point, namely B. Therefore, A, B and C are collinear

Dot/scalar product of two vectors

- The scalar product $\boldsymbol{a}.\boldsymbol{b}$of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by
  $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$where$\theta$ is the angle between the vectors.
- When the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$are perpendicular, $\theta = 90^0$ and $\cos 90^0 = 0$
  Therefore $\boldsymbol{a}.\boldsymbol{b} = 0$
- When the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$is acute, $\cos\theta > 0$ and therefore $\boldsymbol{a}.\boldsymbol{b} > 0$



- When the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$is between $90^0$and 180^0, $\cos\theta < 0$ and therefore $\boldsymbol{a}.\boldsymbol{b} < 0$



- To see how we calculate the scalar product $\boldsymbol{a}.\boldsymbol{b}$, let$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$. Then
  $a.b = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}).(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k})$
  $= a_1b_1 + a_2b_2 + a_3b_3$since$\boldsymbol{i}.\boldsymbol{i} = \boldsymbol{j}.\boldsymbol{j} = \boldsymbol{k}.\boldsymbol{k} = 1$ and$\boldsymbol{i}.\boldsymbol{j} = \boldsymbol{j}.\boldsymbol{k} = \boldsymbol{i}.\boldsymbol{k} = 0$
- We use dot product to find a vector perpendicular to other two vectors

Example 16

Find the scalar product of each of the following pairs of vectors.

(a) $2\boldsymbol{i}+3\boldsymbol{j}\ and\ \ \boldsymbol{i}-6\boldsymbol{j}$(b) $4\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}\ and\ 2\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k}$(c) $\begin{pmatrix}2\\-1\\4\end{pmatrix}\ and\ \begin{pmatrix}-3\\1\\5\end{pmatrix}$

Solution

(a) $(2\boldsymbol{i}+3\boldsymbol{j}).(\boldsymbol{i}-6\boldsymbol{j})=(2\times 1)+(3\times -6)=-16$

(b) $(4\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}).(2\boldsymbol{i}+\boldsymbol{j}-3\boldsymbol{k})=(4\times 2)+(-2\times 1)+(1\times -3)=3$

(c) $\begin{pmatrix}2\\-1\\4\end{pmatrix}.\begin{pmatrix}-3\\1\\5\end{pmatrix}=(2\times -3)+(-1\times 1)+(4\times 5)=13$

Example 17

Find the angle between the vectors $\boldsymbol{a}=2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b}=\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$

Solution

Using $\boldsymbol{a}.\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

$\Rightarrow (2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}).(\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k})=\sqrt{2^2+1^2+1^2}.\sqrt{1^2+(-1)^2+3^2}\cos\theta$

$\Rightarrow (2\times 1)+(1\times -1)+(1\times 3)=\sqrt{6}\sqrt{11}\cos\theta$

$\Rightarrow 4=\sqrt{66}\cos\theta$

$\Rightarrow \cos\theta=\frac{4}{\sqrt{66}}$

$\therefore \theta=60.5^0$

Example 18

Given that the two vectors $\boldsymbol{a}=(3t+1)\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$and $\boldsymbol{b}=(t+3)\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}$are perpendicular, find the possible values of the constant t.

Solution

$\boldsymbol{a}$and$\boldsymbol{b}$ are perpendicular if $\boldsymbol{a}.\boldsymbol{b}=0$

$\Rightarrow \big((3t+1)\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}\big).\big((t+3)\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}\big)=0$

$\Rightarrow (3t+1)(t+3)+(1\times 3)+(-1\times -2)=0$

$\Rightarrow 3t^2+10t+8=0$

$\Rightarrow (3t+4)(t+2)=0$

$\therefore t=-4/3\ \ and\ t=-2$

Example 19

The points A, B, C and D have position vectors $\binom{-2}{3}$, $\binom{3}{8}$, $\binom{7}{6}$and $\binom{7}{-4}$. Show that $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$

Solution

$\Rightarrow \overrightarrow{AC}=\binom{7}{6}-\binom{-2}{3}=\binom{9}{3}$

$\Rightarrow \overrightarrow{BD}=\binom{7}{-4}-\binom{3}{8}=\binom{4}{-12}$

$\Rightarrow \overrightarrow{AC}.\overrightarrow{BD}=\binom{9}{3}.\binom{4}{-12}$

$=(9\times 4)+(3\times -12)$

$= 36 - 36$
$= 0$
$\therefore$ since $\overrightarrow{AC}.\overrightarrow{BD} = 0$, then $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$

Example 20
The vertices of a triangle are P(4, 3), Q(6, 4) and R(5, 8). Find the angle RPQ using vectors.
Solution
$\overrightarrow{PR} = \begin{pmatrix}5\\8\end{pmatrix} - \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$
$\overrightarrow{PQ} = \begin{pmatrix}6\\4\end{pmatrix} - \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$
Let the required angle be θ
$\Rightarrow \overrightarrow{PR}.\overrightarrow{PQ} = |\overrightarrow{PR}||\overrightarrow{PQ}|\cos\theta$
$\Rightarrow \begin{pmatrix}1\\5\end{pmatrix}.\begin{pmatrix}2\\1\end{pmatrix} = \sqrt{1^2+5^2}.\sqrt{2^2+1^2}\cos\theta$
$\Rightarrow (1\times 2) + (5\times 1) = \sqrt{26}.\sqrt{5}\cos\theta$
$\Rightarrow 7 = \sqrt{26\times 5}\cos\theta$
$\therefore \theta = 52.13^0$

Example 21
Given the vectors$\boldsymbol{a} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ find a vector **c** such that it is perpendicular to both **a** and **b**
Solution
Let $\boldsymbol{c} = p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$
$\boldsymbol{a}.\boldsymbol{c} = 0$
$\Rightarrow (3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}).(p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}) = 0$
$\Rightarrow 3p - 2q + r = 0$ ....(1)
$\boldsymbol{b}.\boldsymbol{c} = 0$
$\Rightarrow (\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}).(p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}) = 0$
$\Rightarrow p - 2q + 2r = 0$ ....(2)
$eqn(1) - eqn(2)$
$\Rightarrow 2p - r = \boldsymbol{0}$
$\Rightarrow r = 2p$ ....(3)
Putting equation (3) into equation (1)
$\Rightarrow 3p - 2q + 2p = 0$
$\Rightarrow 2q = 5p$
$\Rightarrow q = \frac{5}{2}p$
$\Rightarrow p: q: r = p: \frac{5}{2}p: 2p$
$\Rightarrow p: q: r = 2: 5: 4$
$\therefore \boldsymbol{c} = 2\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}$

The cross product of vectors

- The cross product of two vector **a** and **b** is denoted as $\boldsymbol{a} \times \boldsymbol{b}$ *or* $\boldsymbol{a} \wedge \boldsymbol{b}$.
- When two vectors **a** and **b** are crossed, it produces a vector perpendicular to both i.e. $n = \boldsymbol{a} \times \boldsymbol{b}$
- The magnitude of the cross product is defined as $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta$ where θ is the angle between **a** and **b**.

Example 22

Given the vectors $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$ find $\boldsymbol{a} \times \boldsymbol{b}$

Solution

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 1 \\ 2 & 4 & 3 \end{vmatrix}$$

$$= \boldsymbol{i}\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= \boldsymbol{i}(6-4) - \boldsymbol{j}(9-4) + \boldsymbol{k}(12-4)$$

$$= 2\boldsymbol{i} - 5\boldsymbol{j} + 8\boldsymbol{k}$$

The area of a triangle

- The area of the triangle is given by $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$ *or* $\frac{1}{2}|\boldsymbol{b} \times \boldsymbol{c}|$ *or* $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{c}|$

Example 23

Find the area of triangle PQR where P is (4, 2, 5), Q is (3, -1, 6) and R is (1, 4, 2)

Solution

$$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

$$\overrightarrow{PR} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix}$$

$$\overrightarrow{PR} \times \overrightarrow{PQ} = \begin{vmatrix} i & -j & k \\ -3 & 2 & -3 \\ -1 & -3 & 1 \end{vmatrix} = -7\boldsymbol{i} + 6\boldsymbol{j} + 11\boldsymbol{k}$$

$$\Rightarrow \left|\overrightarrow{PR} \times \overrightarrow{PQ}\right| = \sqrt{49 + 36 + 121} = \sqrt{206}$$

Area of triangle $= \frac{1}{2}\sqrt{206} = 7.1764\ sq.units$

Exercise 6.1

1. Write down the displacement vector $\overrightarrow{AB}$ for each of the following pairs of points: $A(2,3), B(2,5)$ (b) $A(5,1), B(8,1)$(c)$A(12,5), B(5,4)$ (d) $A\left(\frac{2}{5}, \frac{-3}{7}\right), B\left(\frac{1}{2}, -3\right)$
2. The points A and B have position vectors $\boldsymbol{a}$and $\boldsymbol{b}$. A point C with a position vector $\boldsymbol{c}$lies on AB such that $\frac{AC}{AB} = \lambda$.

3. A, B and C are three collinear points whose position vectors are $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$respectively and $\overrightarrow{AC} = 3\overrightarrow{AB}$. Express $\boldsymbol{c}$in the form $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$
4. A, B and C are three collinear points whose position vectors are $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$respectively and $\overrightarrow{AC} = -2\overrightarrow{AB}$. Express $\boldsymbol{c}$in the form $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$
5. Prove that if **a** and **b** are the position vectors of points A and B, then the position vector of a point P on AB , where $AP: PB = m: n$ is given by $(m+n)\boldsymbol{p} = n\boldsymbol{a} + m\boldsymbol{b}$.
6. A and B are the points (3, 7) and (15, 13) respectively. P is a point on AB such that $\overrightarrow{AP} = s\overrightarrow{AB}$. Write down the vector $\overrightarrow{OP}$ in terms of s. Find the coordinates of P, when
   (a) $s = 3/4$ (b) $s = 3/2$ (c) $s = -2$
   $Ans((a)(12,11.5), (b)(21,16), (c)(-21,-5)$
7. Given that A is the point (2, 5) and that B is the point (10, -1), find the position vector of a point P on AB such that

(a) $\overrightarrow{AP} = \overrightarrow{PB}$ (b) $2\overrightarrow{AP} = \overrightarrow{PB}$ (c) $\overrightarrow{AP} = 4\overrightarrow{AB}$ (d) $AP: PB = 2: 3$(e) $AP: PB = m: n$

$Ans((a)\begin{pmatrix}6\\2\end{pmatrix}(b)\begin{pmatrix}4\frac{2}{3}\\3\end{pmatrix}(c)\begin{pmatrix}34\\-19\end{pmatrix}(d)\begin{pmatrix}5.2\\2.6\end{pmatrix}(e)\frac{1}{m+n}\begin{pmatrix}-7n-m\\3n-15m\end{pmatrix})$

8. A, B and C are three collinear points whose position vectors are **a**, **b** and **c**respectively and $\overrightarrow{AC} = 3\overrightarrow{AB}$.Express **c** in the form$\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$, find the scalars m and n and verify that $m + n = 1$.
   $Ans(-2, 3)$
9. Find numbers m and n such that $m\begin{pmatrix}3\\5\end{pmatrix} + n\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\9\end{pmatrix}$. $Ans(m = 2, n = -1)$
10. Find the magnitude of the following vectors.

(a) $\boldsymbol{v} = 3\boldsymbol{i} + \boldsymbol{j}$(b) $\boldsymbol{a} = -4\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}(c)\boldsymbol{p} = 11\boldsymbol{i} + 5\boldsymbol{k}$

11. Find the magnitude and direction of the following vectors;

(a) $3\boldsymbol{i} + 4\boldsymbol{j}$ (b) $-5\boldsymbol{i} + 12\boldsymbol{j}$ (c) $-10\boldsymbol{j}$ (d)$\boldsymbol{i} - \boldsymbol{j}$

12. Find the unit vector in the direction of $\boldsymbol{v} = 5\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$
13. Find the vector which has magnitude 15 units and is parallel to 16i+12j
14. Find the vector of magnitude 6 units in the direction of the vector $\boldsymbol{i} + \boldsymbol{j}$
15. OABC is a trapezium with $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OC} = \boldsymbol{c}$and $\overrightarrow{CB}$ parallel to and twice as long as $\overrightarrow{OA}$. The point D and E are the mid-points of AB and CB respectively. Find the following vectors in terms of **a** and**c**.

(a) $\overrightarrow{CA}$ (b) $\overrightarrow{AB}$ (c) $\overrightarrow{ED}$.

   Hence show that $\overrightarrow{CA}$ is parallel to and twice as long as $\overrightarrow{ED}$
   $Ans((a) - \boldsymbol{c} + \boldsymbol{a}(b)\boldsymbol{a} + \boldsymbol{c}(c)\frac{1}{2}(\boldsymbol{a} - \boldsymbol{c}))$

16. Triangle OAB has $\overrightarrow{OA} = \boldsymbol{a}$and $\overrightarrow{OB} = \boldsymbol{b}$, C is a point on OA such that $\overrightarrow{OC} = \frac{2}{3}a$. D is the mid-point of AB. When CD is produced meets OB produced at E such that $\overrightarrow{DE} = n\overrightarrow{CD}$and $\overrightarrow{BE} = k\boldsymbol{b}$. Express $\overrightarrow{DE}$ in terms of;

(a) $n, \boldsymbol{a}$ *and* $\boldsymbol{b}$
(b) $k, \boldsymbol{a}$ *and* $\boldsymbol{b}$
(c) Find the values of $n$and k.

17. Show that the points P(4, -3), Q(-3, 4), R(-2, 7) and S(5, 0) are the vertices of a parallelogram PQRS.
18. Given the points A(1, 1), B(5, 4), C(8, 9) and D(0, 3), show that ABCD is a trapezium.
19. If the points A, B and C have position vectors $\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -2 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 7 \\ -1 \end{pmatrix}$ respectively. Show that ABC is a triangle.
20. The points P(2, 3), Q(-11, 8) and R(-4, -5) are vertices of a parallelogram PQRS which has PR as a diagonal. Find the co-ordinates of the vertex S.
21. Given that A(2, 3), B(4, -7) lies on a straight lie find the position vector of the point P that divides AB

(i) Internally in the ratio 1:4
(ii) Externally in the ratio 1:4
(iii) In the ratio 2:-3
(iv) In such a way that AP:AB=2:6
(v) Internally in the ratio -2:3

22. Prove that the vectors $\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ are coplanar.
23. Prove that the vectors $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}$, $\boldsymbol{b} = 5\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{c} = 4\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$ are coplanar.
24. The three points A, B and C have position vectors **a**, **b** and **c** respectively. If $\boldsymbol{c} = 3\boldsymbol{b} - 2\boldsymbol{a}$, Show that A, B and C are collinear.
25. Points P, Q and R have position vectors $\begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 11 \\ 7 \\ 10 \end{pmatrix}$ respectively.
   (a) Find $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.
   (b) Deduce that P,Q and R are collinear and find the ratio $PQ:QR$
26. Given that $\overrightarrow{OP} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ find the co-ordinates of R when point P, Q and R are such that $3\overrightarrow{PQ} = 2\overrightarrow{RQ}$.
27. Given that the points A(2, 13,-5), B(3,y, -3) and C(6, -7, m) are collinear, find m and y.
28. The points A, B and C have coordinates (1, -5, 6), (3, -2, 10) and (7, 4, 18) respectively. Show that A, B and C are collinear.
29. Given that A(2, 13, -5), B(3, β, -3) and C(6, -7, α) are collinear, find the constants β and α. $Ans(\beta = 8, \alpha = 3)$
30. Given that $\overrightarrow{AB} = \alpha\boldsymbol{i} + 6\boldsymbol{j} + 4\boldsymbol{k}$, $\overrightarrow{BC} = 4\boldsymbol{i} + \beta\boldsymbol{j} - 3\boldsymbol{k}$ and $\overrightarrow{AC} = -3\boldsymbol{i} + \gamma\boldsymbol{k}$, find the values of the constants α, β and γ. $Ans(\alpha = -7, \beta = -6, \gamma = 1)$
31. Given that the vectors $a\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $2a\boldsymbol{i} + a\boldsymbol{j} - 4\boldsymbol{k}$ are perpendicular, find the values of a.
32. If the points A, B and C have position vectors $\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}$, $3\boldsymbol{i} - 2\boldsymbol{j} + 9\boldsymbol{k}$ and $8\boldsymbol{i} + 7\boldsymbol{j} - \boldsymbol{k}$ respectively find the angle ABC giving your answer to the nearest degree.

33. The points P(4, -6, 1), Q(2, 8, 4) and R(3, 7, 14) lie in the same plane . Find the angle formed between **PQ** and **QR.**

34. Find the angle between each of the following pairs of vectors

(a) $3\boldsymbol{i}-4\boldsymbol{j}\ and\ 12\boldsymbol{i}+5\boldsymbol{j}(b)\binom{2}{-1}\ and\ \binom{6}{3}$ (c) $2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}\ and\ 4\boldsymbol{i}-3\boldsymbol{j}+12\boldsymbol{k}$

$Ans((a)75.7^0(b)53.1^0(c)119.2^0\ )$

35. Given$\boldsymbol{a}=4\boldsymbol{i}+5\boldsymbol{j},\ \boldsymbol{b}=\lambda\boldsymbol{i}-8\boldsymbol{j}\ \boldsymbol{c}=\boldsymbol{i}+\mu\boldsymbol{j}$

(a) Find the value of the constant λ given that **a** and **b** are perpendicular

(b) Find the value of the constant μ given that **a** and**c** are parallel.

$Ans(\lambda=10,=1\frac{1}{4})$

36. Given that $\begin{pmatrix}\lambda\\2+\lambda\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\3\\4-\lambda\end{pmatrix}$ are perpendicular vectors, find the value of the constant λ. $Ans(\lambda=18)$

37. Find the unit vector perpendicular both vectors $4\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}\ and\ 2\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k}.$

$Ans\left\{\frac{1}{9\sqrt{5}}(7\boldsymbol{i}+10\boldsymbol{j}+16\boldsymbol{k})\right\}$

38. Evaluate $(2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k})\times(7\boldsymbol{i}+4\boldsymbol{j}+2\boldsymbol{k})Ans(10\boldsymbol{i}-11\boldsymbol{j}-13\boldsymbol{k})$

39. Evaluate $|\overrightarrow{AB}\times\overrightarrow{CD}|$, where A is (6, -3, 0), B is (3, --7, 1), C is (3, 7, -1) and D is (4, 5, -3).$Ans(\sqrt{15})$

40. Show that the vectors $\boldsymbol{i}-2\boldsymbol{k},-2\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}$ and $-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$ form a triangle. Hence find the area of the triangle.

6.2 Lines in two and Three Dimensions

Vector equations of lines

- The general equation of a line through the point A and in the direction of **b** is $\boldsymbol{r}=\boldsymbol{a}+t\boldsymbol{b}$where a is the position vector of A, and each value of the parameter t corresponds to a point on the line.

Example 24

(a)Find the equation of the line through the point (2, 4, 5) in the direction $-2\boldsymbol{i}+3\boldsymbol{j}+8\boldsymbol{k}$

(b) Find p and q so that the point (p, 10, q) lies on the line.

Solution

(a) The equation of the line is $r=\begin{pmatrix}2\\4\\5\end{pmatrix}+t\begin{pmatrix}-2\\3\\8\end{pmatrix}$

(b) If the point (p, 10, q) lies on the line, then for some t we have

$$\begin{pmatrix}2\\4\\5\end{pmatrix}+t\begin{pmatrix}-2\\3\\8\end{pmatrix}=\begin{pmatrix}p\\10\\q\end{pmatrix}$$

Considering these coordinates, we have

For i; $2 - 2t = p$……………..(1)
For j; $4 + 3t = 10$……………..(2)
For k; $5 + 8t = q$………………..(3)
From (2) $t = 2$
Substituting $t = 2$ in (1) and (3), we get
$p = -2$ and $q = 21$

- Let **a** and **b** be the position vectors of two point A and B with respect to the origin O. Let **r** be the position vector of a point P on the line AB

We have
$\overrightarrow{OP} = \overrightarrow{OA} + t\overrightarrow{AB}$ or $\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$

Example 25

Find the vector equation of the line passing through A(1, 3, 2) and B(0, -1, 4). Does the point P(-2, 9, 1) lie on the line AB?

Solution

The position vectors of A and B are $a = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$

The vector equation of the line is given by

$\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t\left(\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}\right)$$

$$r = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$$

This could be written as $\boldsymbol{r} = (1 - t)\boldsymbol{i} + (3 - 4t)\boldsymbol{j} + (2 + 2t)\boldsymbol{k}$.

If the point P(-2, 9, 1) lie on the line AB, there will exist unique value of t for which

$$\begin{pmatrix} -2 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$$

If $-2 = 1 - y, then\ t = 3$. However we see that $t = 3$ does not satisfy $9 = 3 - 4t$.

Therefore the point P(-2 9, 1) does not lie on the line AB.

- When given three points on the line i.e. A, B and C, the vector equation of the line is given by $r = \overrightarrow{OA} + \mu\overrightarrow{BC}\ or\ \boldsymbol{r} = \boldsymbol{a} + \mu(\boldsymbol{c} - \boldsymbol{b})$

Qn. Find the vector equation of the line passing through the points A(1, 0, 1), B(2, -1, 5) and C(4, 1, 0)

- Whenever they say find the equation of the line, it means the vector equation.

The Parametric equation of a line

- These can expressed as seen in the following example;

Example 26

Express $\boldsymbol{r} = 2\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k} + \mu(-\boldsymbol{i} - 2\boldsymbol{k})$in parametric form

Solution

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$$

$x = 2 - \mu$ ……………………….(1)

$y = 2$ …………………………….(2)

$z = 5 - 2\mu$ ……………………..(3)

∴equations (1), (2) and (3) are the parametric equations of the line.

The Cartesian equations of a line

- To find the Cartesian equation of a line which passes though the point (x_1, y_1, z_1) in the direction $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$, we use the vector equation

  $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$

  Hence, we obtain the vector equation of this line as

  $$r = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

- Let the general vector $\boldsymbol{r}$be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, which gives

  $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

- Using the **i**, **j**, **k** components, we have the parametric equations

  $x = x_1 + tl$, $y = y_1 + tm$ and $z = z_1 + tn$

- Finding t from each of these components, we get

  $t = \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

- Hence the Cartesian equation of a straight line which passes through the point (x_1, y_1, z_1) in the direction $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is $\boldsymbol{\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}}$

  Note: we equate the numerators to zero in order to get the point on the line and the denominators make the directional vector if and only if the coefficients of x, y and z are +1.

Example 27

Find the Cartesian equation of the line PQ, where P is (2, 1, 7) and Q is (3, 8, 4)

Solution

$\Rightarrow \overrightarrow{PQ} = \begin{pmatrix}3\\8\\4\end{pmatrix} - \begin{pmatrix}2\\1\\7\end{pmatrix} = \begin{pmatrix}1\\7\\-3\end{pmatrix}$

Hence, given that the line passes through P(2, 1,7), its vector equation is $\boldsymbol{r} = \begin{pmatrix}2\\1\\7\end{pmatrix} + t\begin{pmatrix}1\\7\\-3\end{pmatrix}$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\1\\7\end{pmatrix} + t\begin{pmatrix}1\\7\\-3\end{pmatrix}$

The its Cartesian equation is given by $\frac{x-2}{1} = \frac{y-1}{7} = \frac{z-7}{-3}$.

Qn. For the line through (4, 7, -1) in the direction $2\boldsymbol{i} - 3\boldsymbol{j} - 5\boldsymbol{k}$, find;

(a) its vector equation

(b) its parametric equations

(c) its Cartesian equation.

Example 28

Find the vector equation of the line $\frac{x-3}{4} = \frac{1-y}{2} = \frac{2z+7}{5}$

Solution

We always start by rearranging the Cartesian equation in the form $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

Which in this case gives $\frac{x-3}{4} = \frac{y-1}{-2} = \frac{z+\frac{7}{2}}{\frac{5}{2}}$

Therefore the vector equation of the line is $r = \begin{pmatrix}3\\1\\-\frac{7}{2}\end{pmatrix} + t\begin{pmatrix}4\\-2\\\frac{5}{2}\end{pmatrix}$

Angle between lines

- This is obtained from $\boldsymbol{b_1}.\boldsymbol{b_2} = |\boldsymbol{b_1}||\boldsymbol{b_2}|\cos\theta$ where $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ are the directional/ parallel vectors to the two lines and θ is the angle required.

Example 29

Find the angle between the two lines $\frac{x-3}{4} = \frac{y-5}{2} = \frac{z-8}{-1}$ and $\boldsymbol{r} = \begin{pmatrix}3\\-1\\-2\end{pmatrix} + t\begin{pmatrix}-7\\4\\3\end{pmatrix}$

Solution

The directional vectors are $\boldsymbol{b_1} = \begin{pmatrix}4\\2\\-1\end{pmatrix}$ and $\boldsymbol{b_2} = \begin{pmatrix}-7\\4\\3\end{pmatrix}$

Let θ be the required angle

$$\Rightarrow \cos\theta = \frac{\begin{pmatrix}4\\2\\-1\end{pmatrix}.\begin{pmatrix}-7\\4\\3\end{pmatrix}}{\sqrt{4^2+2^2+(-1)^2}.\sqrt{(-7)^2+4^2+3^2}}$$
$$= \frac{-28+8-3}{\sqrt{21}.\sqrt{74}}$$
$$= -\frac{23}{\sqrt{1554}}$$

The minus sign indicates that the angle between the two lines is obtuse. However, the angle between the two lines would normally be taken to be acute. Therefore, the angle between the two lines is

$\cos^{-1}\left(\frac{23}{\sqrt{1554}}\right) = 54.3^0$

Shortest distance of a point from a line

- In this we can be asked to find the shortest distance (perpendicular distance), the equation of the perpendicular and the coordinates of the foot of the perpendicular from a point to the line.

Example 30

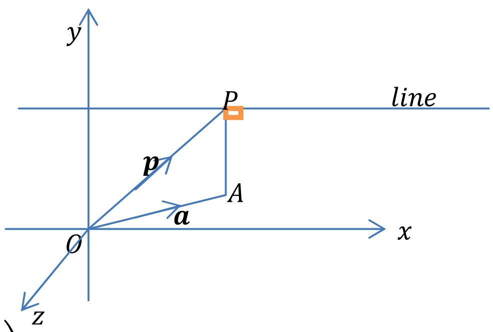
Given the point A(4, -3, 10) and the line $r = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}3\\-1\\2\end{pmatrix}$

Find the;

(a) Coordinates of the foot of the perpendicular from A to the line.
(b) The perpendicular distance from A to the line.
(c) The equation of the perpendicular.

Solution

(a) Let P be the point where the perpendicular from A meets the line and P has the position vector $\boldsymbol{p} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu_1\begin{pmatrix}3\\-1\\2\end{pmatrix} = \begin{pmatrix}1+3\mu_1\\2-\mu_1\\3+2\mu_1\end{pmatrix}$



$$\Rightarrow \overrightarrow{PA} = \begin{pmatrix}4\\-3\\10\end{pmatrix} - \begin{pmatrix}1+3\mu_1\\2-\mu_1\\3+2\mu_1\end{pmatrix}$$
$$= \begin{pmatrix}3-3\mu_1\\-5+\mu_1\\7-2\mu_1\end{pmatrix}$$

But $\overrightarrow{PA}$ is perpendicular to L and L is parallel to $\begin{pmatrix}3\\-1\\2\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}3-3\mu_1\\-5+\mu_1\\7-2\mu_1\end{pmatrix}.\begin{pmatrix}3\\-1\\2\end{pmatrix} = 0$$
$$\Rightarrow 9-9\mu_1+5-\mu_1+14-4\mu_1 = 0$$
$$\Rightarrow \mu_1 = 2$$

$$p = \begin{pmatrix} 1 + 3\mu_1 \\ 2 - \mu_1 \\ 3 + 2\mu_1 \end{pmatrix} = \begin{pmatrix} 1 + 3(2) \\ 2 - 2 \\ 3 + 2(2) \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 7 \end{pmatrix}$$

∴the coordinates of the foot are P(7, 0, 7)

(b) $\overrightarrow{PA} = \begin{pmatrix} 3 - 3\mu_1 \\ -5 + \mu_1 \\ 7 - 2\mu_1 \end{pmatrix} = \begin{pmatrix} 3 - 3(2) \\ -5 + (2) \\ 7 - 2(2) \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}$

The perpendicular distance $= |\overrightarrow{PA}| = \sqrt{(-3)^2 + (-3)^2 + 3^2} = 3\sqrt{3} = 5.1962\ units$

(c) The equation of the perpendicular is given by $\boldsymbol{r} = \overrightarrow{OA} + \lambda\overrightarrow{PA}$

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}$$

Intersection of two lines

- For two lines to intersect there must exist some point (x, y, z) that lies on both lines i.e. we equate the two lines.

Example 31

Two lines l and m have vector equations $r_l = (2 - 3\lambda)\boldsymbol{i} + (1 + \lambda)\boldsymbol{j} + (4\lambda)\boldsymbol{k}$and $r_m = (-1 + 3\mu)\boldsymbol{i} + (3)\boldsymbol{j} + (7 - \mu)\boldsymbol{k}$respectively. Find;

(a) the position vector of their common point.

(b) the angle between the lines

Solution

(a) $\boldsymbol{r}_l = \begin{pmatrix} 2 - 3\lambda \\ 1 + \lambda \\ 4\lambda \end{pmatrix}$ and $\boldsymbol{r}_m = \begin{pmatrix} -1 + 3\mu \\ 3 \\ 7 - \mu \end{pmatrix}$

At the point common to $l$and m, we have

$\boldsymbol{r}_l = \boldsymbol{r}_m$

$$\Rightarrow \begin{pmatrix} 2 - 3\lambda \\ 1 + \lambda \\ 4\lambda \end{pmatrix} = \begin{pmatrix} -1 + 3\mu \\ 3 \\ 7 - \mu \end{pmatrix}$$

Equating **i**, **j** and **k** to coefficients, we get

i: $2 - 3\lambda = -1 + 3\mu$……………(1)

j: $1 + \lambda = 3$…………………………(2)

k: $4\lambda = 7 - \mu$……………………….(3)

From (1) $\lambda = 2$ substituting in (1) gives

$2 - 3(2) = -1 + 3\mu\ ;\ \mu = -1$

We notice that $\lambda = 2, \mu = -1$ also satisfies (3). So, at the common point we have

$$\boldsymbol{r}_l = \begin{pmatrix} 2 - 3(2) \\ 1 + (2) \\ 4(2) \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix}$$

∴the position vector of the common point is $-4\boldsymbol{i} + 3\boldsymbol{j} + 8\boldsymbol{k}$.

(b) We know that $\boldsymbol{r_l} = \begin{pmatrix}2\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}-3\\1\\4\end{pmatrix}$ and $\boldsymbol{r}_m = \begin{pmatrix}-1\\3\\7\end{pmatrix} + \mu\begin{pmatrix}3\\0\\-1\end{pmatrix}$

the directional vectors are $\boldsymbol{b}_1 = \begin{pmatrix}-3\\1\\4\end{pmatrix}$ and $\boldsymbol{b}_2 = \begin{pmatrix}3\\0\\-1\end{pmatrix}$

let the required angle be α

$$\Rightarrow \cos\alpha = \frac{\begin{pmatrix}-3\\1\\4\end{pmatrix}.\begin{pmatrix}3\\0\\-1\end{pmatrix}}{\sqrt{(-3)^2+1^2+4^2}.\sqrt{3^2+(-1)^2}}$$

$$= \frac{-9-4}{\sqrt{26}.\sqrt{10}}$$

$$= -\frac{13}{\sqrt{260}}$$

$$\alpha = 143.7^0$$

Example 32

The lines L_1 and L_2 have Cartesian equations $\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1}$ and $\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}$ respectively. Show that L_1 and L_2 intersect and find the coordinates of the point of intersection.

Solution

Let $\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1} = t$ and $\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1} = s$

$$\Rightarrow \begin{pmatrix}t\\-2+2t\\5-t\end{pmatrix} = \begin{pmatrix}1-s\\-3-3s\\4+s\end{pmatrix}$$

Equating **i**, **j** and **k** to coefficients, we get

i: $t = 1-s$; $t+s = 1$……………………………(1)

j: $-2+2t = -3-3s$; $2t+3s = -1$………(2)

k: $5-t = 4+s$; $-t-s = -1$…………………..(3)

2xeqn(1)-eqn(2)

$-s = 3$; $s = -3$

Putting $s = -3$ in (1) gives

$t-3 = 1$; $t = 4$

Putting $s = -3$ *and* $t = 4$ in (3) we get

$-4-(-3) = -1 = RHS$

∴the lines L_1 and L_2 do intersect

$$\Rightarrow \begin{pmatrix}t\\-2+2t\\5-t\end{pmatrix} = \begin{pmatrix}4\\-2+2(4)\\5-4\end{pmatrix} = \begin{pmatrix}4\\6\\1\end{pmatrix}$$

∴the point of intersection is (4, 6, 1)

Parallel and skew lines

- Two lines are said to be parallel if either they have the same directional vector or their directional vectors are parallel.

- Two lines are said to be skew if they neither meet nor parallel.

Exercise 6.2

1. Find the vector equation of the straight line that is parallel to the vector $2\boldsymbol{i}-\boldsymbol{j}$ and which passes through the point with the position vector $3\boldsymbol{i}+2\boldsymbol{j}$
2. Find the vector equation of the line;

(a) through A(2, -7, 5) in the direction $3\boldsymbol{i}+4\boldsymbol{j}-7\boldsymbol{k}$.

(b) through Q(-8, 1, -3) in the direction $\boldsymbol{i}+3\boldsymbol{j}-7\boldsymbol{k}$

3. Find the Cartesian equation of the line through each pair of points

(a) A(4, 8, -2) and B(1, -3, 4)

(b) P(1, 7, -2) and Q(-3, 4, -3)

4. Points A and B have position vectors $2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$ and $5\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$ respectively. Find the vector equation of the line AB.
5. Find the equation of the line which passes through the points A, B and C given that the position vectors of A, B and C are $\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}2\\7\end{pmatrix}$ and $\begin{pmatrix}3\\9\end{pmatrix}$.
6. Show that the point with position vector$\boldsymbol{i}+2\boldsymbol{j}$lie on the line L with the vector equation $\boldsymbol{r}=4\boldsymbol{i}-\boldsymbol{j}+\beta(\boldsymbol{i}-\boldsymbol{j})$.
7. Find the equation of the line passing through the point Q(1, 3, -2) and is perpendicular to the plane containing the vectors $-3\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$ and $6\boldsymbol{i}-5\boldsymbol{j}+2\boldsymbol{k}$
8. Find the equation of the line through the points A(1, 2, 3) and B(4, 4, 4) and find the coordinates of the point where this line meets the plane $z=0$. $Ans(-8,-4,0)$
9. Given the equation of the line in the form $\frac{x-2}{3}=\frac{y-4}{5}=\frac{z-7}{2}$ express the equation in the form $\boldsymbol{r}=\boldsymbol{a}+t\boldsymbol{b}$and show that the line passes through the point (8, 14, 11)
10. Find the unit vector which is parallel to the line $\frac{x-1}{3}=\frac{y-2}{4}=\frac{z-7}{12}$
11. Show that the equations $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\3\\1\end{pmatrix}+m\begin{pmatrix}4\\6\\-2\end{pmatrix}$ and $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}10\\15\\-3\end{pmatrix}+m\begin{pmatrix}-2\\-3\\1\end{pmatrix}$ represent the same line.
12. Find the acute angle between the lines with vector equations $\boldsymbol{r}=2\boldsymbol{i}+3\boldsymbol{j}+\lambda(\boldsymbol{i}+2\boldsymbol{j})$and $\boldsymbol{r}=3\boldsymbol{i}-2\boldsymbol{j}+\lambda(4\boldsymbol{i}-\boldsymbol{j})$, giving your answer to the nearest degree. $Ans(77^0)$
13. Find the acute angle between the lines the equations

(a) $\boldsymbol{r}=3\boldsymbol{i}+4\boldsymbol{j}-7\boldsymbol{k}+t(2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k})$ and$\boldsymbol{r}=-2\boldsymbol{i}+7\boldsymbol{j}+2\boldsymbol{k}+t(3\boldsymbol{i}+5\boldsymbol{j}-3\boldsymbol{k})$

(b) $\boldsymbol{r}=\begin{pmatrix}1\\2\\4\end{pmatrix}+t\begin{pmatrix}7\\-2\\3\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}-2\\3\\11\end{pmatrix}+s\begin{pmatrix}0\\2\\1\end{pmatrix}$

14. Find the equation of the perpendicular line form $\boldsymbol{OA}=\begin{pmatrix}2\\-1\\4\end{pmatrix}$ onto the line

$$r = \begin{pmatrix}1\\0\\2\end{pmatrix} + \mu\begin{pmatrix}2\\1\\2\end{pmatrix}$$

15. Find the distance between the pair of parallel lines listed below.

(a) $\boldsymbol{r} = \begin{pmatrix}2\\0\\3\end{pmatrix} + t\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}1\\-1\\4\end{pmatrix} + s\begin{pmatrix}1\\-1\\2\end{pmatrix}$

(b) $\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z-3}{2}$ and $\frac{x+1}{1} = \frac{y-3}{-1} = \frac{z-1}{2}$

16. Find the perpendicular distance from the point A, position vector $4\boldsymbol{i} + 5\boldsymbol{j}$, to the line L, vector equation $\boldsymbol{r} = -3\boldsymbol{i} + \boldsymbol{j} + \lambda(\boldsymbol{i} + 2\boldsymbol{j})$. $Ans(2\sqrt{5})$

17. Show that the line $\boldsymbol{r} = \begin{pmatrix}2\\5\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix}3\\-3\end{pmatrix} + \lambda\begin{pmatrix}3\\2\end{pmatrix}$ are perpendicular and find the position vector of their point of intersection. $Ans\begin{pmatrix}6\\-1\end{pmatrix}$

18. For each of the following parts, find the perpendicular distance from the given point to the given line.

(a) The point with position vector $\boldsymbol{i} + 5\boldsymbol{j}$ to the line $\boldsymbol{r} = -2\boldsymbol{i} + \boldsymbol{j} + \lambda(2\boldsymbol{i} + \boldsymbol{j})$

(b) The point with position vector $2\boldsymbol{i} - \boldsymbol{j}$ to the line $\boldsymbol{r} = -2\boldsymbol{j} + \lambda(4\boldsymbol{i} - 3\boldsymbol{j})$

(c) The point with position vector $\begin{pmatrix}5\\2\end{pmatrix}$ to the line $\boldsymbol{r} = \begin{pmatrix}6\\-5\end{pmatrix} + \lambda\begin{pmatrix}1\\3\end{pmatrix}$

19. The lines L_1 and L_2 intersect at A and the lines L_1 and L_3 intersect at B. Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$ given that L_1, L_2 and L_3 have vector equations as follows:

$L_1: \boldsymbol{r} = \boldsymbol{i} + 3\boldsymbol{j} + \lambda(\boldsymbol{i} + \boldsymbol{j})$

$L_2: \boldsymbol{r} = \boldsymbol{i} + \mu(5\boldsymbol{i} + 2\boldsymbol{j})$

$L_3: \boldsymbol{r} = 4\boldsymbol{i} + 3\boldsymbol{j} + \eta(2\boldsymbol{i} - \boldsymbol{j})$

$Ans(6\boldsymbol{i} + 6\boldsymbol{j}, 6\sqrt{2}units)$

20. Referred to a fixed origin O, the points P, Q and R have position vector $2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, 5\boldsymbol{j} + 3\boldsymbol{k}$ *and* $5\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$ respectively.

(a) Find in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$, an equation of the line PQ.

(b) Show that the point S with position vector $4\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ lies on PQ.

(c) Show that the line PQ and RS are perpendicular.

(d) Find the size of $\angle PQR$, giving your answer to 0.1^0.

21. The vector equations of three lines are stated below

$L_1: \boldsymbol{r} = 17\boldsymbol{i} + 2\boldsymbol{j} - 6\boldsymbol{k} + \lambda(-9\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k})$

$L_2: \boldsymbol{r} = 2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k} + \mu(6\boldsymbol{i} + 7\boldsymbol{j} - \boldsymbol{k})$

$L_3: \boldsymbol{r} = 2\boldsymbol{i} - 12\boldsymbol{j} - \boldsymbol{k} + \eta(-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$

State which of the lines (a) are parallel to each other

(b) intersect with each other

(c) are not parallel and do not intersect (i.e. are skew)

$Ans((a)1\&3, (b)2\&3\ (c)1\&2)$

22. Find the vector equation of the line with parametric equations $x = 2 + 3\lambda, y = 5 - 2\lambda, z = 4 - \lambda$.

23. With respect to a fixed origin O the lines L1 and L2 are given by the equations
L1: $\boldsymbol{r} = 2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k} + \lambda(-2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$
L2: $\boldsymbol{r} = -6\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k} + \mu(5\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k})$
Where λ *and* μ are scalar parameters.
(a) Show that L1 and L2 meet and find the position vector of their point of intersection.
(b) Find, to the nearest 0.1^0, the acute angle between L1 and L2.

24. Points D, E and F have position vectors $\boldsymbol{i} - 2\boldsymbol{j}$, $4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $7\boldsymbol{i} - 8\boldsymbol{j} - 4\boldsymbol{k}$ respectively. Find which of these points lie on the line with vector equation $\boldsymbol{r} = (2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}) + \lambda(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$.

25. Two lines have vector equations $\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k} + \lambda(4\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k})$ and $\boldsymbol{r} = 2\boldsymbol{i} + 6\boldsymbol{j} - 13\boldsymbol{k} + \lambda(-\boldsymbol{3i} + \boldsymbol{j} + a\boldsymbol{k})$ where λ and μ are scalar parameters and a is a constant. Given that these two lines intersect, find the position vector of the point f intersection and the value of a.

26. The two lines $\frac{x+11}{4} = \frac{y+2}{1} = \frac{z+6}{-2}$ and $\frac{x-6}{5} = \frac{y-5}{4} = \frac{z+20}{-8}$ intersect. Find the coordinates of the point of intersection.

27. The lines L_1 and L_2 have vector equations $\boldsymbol{r} = 8\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k} + \lambda(-4\boldsymbol{i} + \boldsymbol{j})$ and $\boldsymbol{r} = -2\boldsymbol{i} + 8\boldsymbol{j} - \boldsymbol{k} + \mu(\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k})$ respectively. Show that L_1 and L_2 intersect and find the position vector of the point of intersection. $Ans(-4\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$

28. Lines L_1 and L_2 have vector equations $r = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ and $r = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ respectively. Show that L_1 and L_2 intersect and find the position vector of the point of intersection. $Ans \begin{pmatrix} 9 \\ 3 \\ 0 \end{pmatrix}$

29. For each of the following pairs of lines state whether the two lines intersect and. For those that do, give the coordinates of the point of intersection.
(a) $\frac{x-1}{2} = \frac{y+2}{1} = \frac{z-3}{-1}$; $\frac{x+1}{-2} = y - 3 = \frac{z-7}{2}$
(b) $x - 2 = \frac{y+3}{4} = \frac{z-5}{2}$; $\frac{x-1}{-1} = \frac{y-8}{1} = \frac{z-3}{-1}$
(c) $\frac{x-2}{5} = \frac{y-3}{-3} = \frac{z+1}{2}$; $\frac{x-9}{-3} = \frac{y-2}{5} = \frac{z-2}{-1}$
(d) $\frac{x-1}{1} = \frac{y+1}{3} = \frac{z-2}{-1}$; $\frac{x+3}{-2} = \frac{y-8}{1} = \frac{z+2}{-1}$
$Ans\big((a)(5, \quad 0, \quad 1)(b)(4,5,9), (c)12, -3, 3\big), (d) lines\ do\ not\ intersect)$

30. Show that the lines $\boldsymbol{r_1} = (6 - 2\lambda)\boldsymbol{i} + (\lambda - 5)\boldsymbol{j}$, $\boldsymbol{r_2} = (\mu)\boldsymbol{i} + 3(1 - \mu)\boldsymbol{j}$ and $\boldsymbol{r_3} = (5 - v)\boldsymbol{i} + (2v - 9)\boldsymbol{j}$ are concurrent, and find the position vector of the point of intersection. $Ans(2\boldsymbol{i} - 3\boldsymbol{j})$

31. The points A, B and C have position vectors $4\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ and $3\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$, respectively.
(a) Given that the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is θ,

(i) find the value of $\cos\theta$

(ii) deduce that $\sin\theta = \frac{\sqrt{7}}{3}$

(b) Hence show that the perpendicular distance from the point C to the line AB is $\sqrt{6}$.

32. Points A, B and C have position vectors $-\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$, $5\boldsymbol{i} + 6\boldsymbol{j} - 4\boldsymbol{k}$ and $4\boldsymbol{i} + 7\boldsymbol{j} + 5\boldsymbol{k}$ respectively. P is the point on AB such that $\overrightarrow{AP} = \lambda\overrightarrow{AB}$

(a) Find $\overrightarrow{AB}$

(b) Find $\overrightarrow{CP}$

(c) By considering the scalar product $\overrightarrow{AB}.\overrightarrow{CP}$ find the position vector of the point on the line AB which is closest to C.

(d) Deduce that the perpendicular distance from the point C to the line AB is $3\sqrt{3}$.

$Ans((a)(6\boldsymbol{i} + 3\boldsymbol{j} - 9\boldsymbol{k}), (b)(6\lambda - 5)\boldsymbol{i} + (3\lambda - 4)\boldsymbol{j} - (9\lambda)\boldsymbol{k},$
$(c)\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k})$

6.3 Planes

Equation of a plane

Equation in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + s\boldsymbol{c}$

- The position vector of any point on a plane can be expressed in terms of;

(i) **a**, the position vector of a point on the plane, and

(ii) **b** and **c**, which are two non-parallel vectors in the plane.

The equation in the form $\boldsymbol{r}.\boldsymbol{n} = d$

- Given n is a vector perpendicular to the plane, we have

$\boldsymbol{r}.\boldsymbol{n} = (\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + s\boldsymbol{c}).\boldsymbol{n}$

$= \boldsymbol{a}.\boldsymbol{n} + t\boldsymbol{b}.\boldsymbol{n} + s\boldsymbol{c}.\boldsymbol{n}$

Since b and c are perpendicular to n, $\boldsymbol{b}.\boldsymbol{n} = \boldsymbol{c}.\boldsymbol{n} = 0$. Hence, we have

$\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{n}.\boldsymbol{a}$

- The vector equation (in scalar product form) of the plane is $\boldsymbol{r}.\boldsymbol{n} = d$ where d is a constant which determines the position of the plane.

Note the following:

(i) If $\boldsymbol{n}$ is a unit vector, then d is the perpendicular distance of the plane from the origin.

(ii) When d has the **same sign** for two planes, these planes are on the **sameside** of the origin.

(iii) When d has **opposite signs** for two planes, these planes are on **opposite sides** of the origin.

(iv) Planes are said to be parallel if they have the same normal vector $\boldsymbol{n}$.

Parametric equation of the plane

- The equation $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b} + s\boldsymbol{c}$ is also known as the parametric vector equation of the plane.

- Therefore if it is expressed in the form $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+t\begin{pmatrix}a\\b\\c\end{pmatrix}+s\begin{pmatrix}p\\q\\w\end{pmatrix}$ then the equations $x=x_1+ta+sp,\ y=y_1+tb+sq$ and $z=z_1+tc+sw$ are the real parametric equations of the plane.

Cartesian equations of a plane

- Taking $\boldsymbol{r}.\boldsymbol{n}=d$ and replacing $\boldsymbol{r}$by $x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, gives the equation of the plane as $\begin{pmatrix}x\\y\\z\end{pmatrix}.\boldsymbol{n}=d.$
- If we let $\boldsymbol{n}=a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}$, then the Cartesian equation becomes $\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}a\\b\\c\end{pmatrix}=d$ or $ax+by+cz=d$

Example 33

Find the equation of the plane through (3, 2, 7) which is perpendicular to the vector $\begin{pmatrix}1\\-5\\8\end{pmatrix}$, giving its equation

(a) in vector form and

(b) in Cartesian form.

Solution

(a) Using $\boldsymbol{r}.\boldsymbol{n}=\boldsymbol{a}.\boldsymbol{n}$, we have

$$r.\begin{pmatrix}1\\-5\\8\end{pmatrix}=\begin{pmatrix}3\\2\\7\end{pmatrix}.\begin{pmatrix}1\\-5\\8\end{pmatrix}$$

Hence, the vector equation of the plane is $\boldsymbol{r}.\begin{pmatrix}1\\-5\\8\end{pmatrix}=49$

(b) Replacing $\boldsymbol{r}\ by\ x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, we get

$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}1\\-5\\8\end{pmatrix}=49$$

Therefore the Cartesian equation is $x-5y+8z=49$.

Note: A plane is defined by;

(i) a vector perpendicular to the plane and

(ii) a point on the plane.

Example 34

Find the unit vector perpendicular to the plane $2x+3y-7z=11$

Solution

The vector $\begin{pmatrix}a\\b\\c\end{pmatrix}$ is perpendicular to the plane $ax+by+cz=d$

Therefore the perpendicular vector to the plane is $\begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}$

The magnitude of this vector is $\sqrt{2^2 + 3^2 + (-7)^2} = \sqrt{62}$.

The unit vector perpendicular to the given plane is $\begin{pmatrix} \frac{2}{\sqrt{62}} \\ \frac{3}{\sqrt{62}} \\ \frac{-7}{\sqrt{62}} \end{pmatrix}$

Example 35

Find the Cartesian equation of the plane with parametric vector equation

$r = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

Solution

We eliminate λ and μ in order to form the Cartesian equation.

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

$x = 1 + \lambda$ ………………….(1)

$y = 2 + \lambda + 2\mu$ ……………(2)

$z = -1 + 2\lambda - \mu$ …………….(3)

From (1) $\lambda = x - 1$

Putting λ in (2) gives

$y = 2 + (x - 1) + 2\mu;\ \mu = \frac{y-x-1}{2}$

Putting λ and μ in (3) gives

$z = -1 + 2(x - 1) - \frac{(y-x-1)}{2}$

$\Rightarrow 2z = -2 + 4x - 4 - y + x + 1$

$\therefore 5x - y - 2z = 5$is the Cartesian equation.

Example 36

Find the equation of the plane through A(1, 4, 6), B(2, 7, 5) and C(-3, 8, 7)

Solution

Method 1

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} -3 \\ 8 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix}$

Using the equation $r = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix}$

$x = 1 + \lambda - 4\mu$ ………………..(1)
$y = 4 + 3\lambda + 4\mu$ ………………(2)
$z = 6 - \lambda + \mu$ ……………………(3)
Adding (1) and (3)
$x + z = 7 - 3\mu;\ \mu = \frac{x+z-7}{-3}$
Putting μ in (3)
$\lambda = 6 - z + \frac{x+z-7}{-3}$
$\lambda = \frac{-18+3z+x+z-7}{-3} = \frac{x+4z-25}{-3}$
Putting $\lambda$and μ in (2)
$\Rightarrow y = 4 + 3\left(\frac{x+4z-25}{-3}\right) + 4\left(\frac{x+z-7}{-3}\right)$

$\Rightarrow -3y = -12 + 3x + 12z - 75 + 4x + 4z - 28$
$\therefore 7x + 3y + 16z = 115$

Method 2

With $\overrightarrow{AB} = \begin{pmatrix}1\\3\\-1\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}-4\\4\\1\end{pmatrix}$

$\Rightarrow n = \overrightarrow{AB} \times \overrightarrow{AC}$

$= \begin{vmatrix} i & -j & k \\ 1 & 3 & -1 \\ -4 & 4 & 1 \end{vmatrix}$

$= 7\boldsymbol{i} + 3\boldsymbol{j} + 16\boldsymbol{k}$

Using $\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}7\\3\\16\end{pmatrix} = \begin{pmatrix}1\\4\\6\end{pmatrix}.\begin{pmatrix}7\\3\\16\end{pmatrix}$

$\Rightarrow 7x + 3y + 16z = 7 + 13 + 96$
$\therefore 7x + 3y + 16z = 115$

Note: instead of **a** we can take **b** or **c** in the formula $\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$

Method 3

With $\overrightarrow{AB} = \begin{pmatrix}1\\3\\-1\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}-4\\4\\1\end{pmatrix}$

We let $\boldsymbol{n} = \begin{pmatrix}p\\q\\r\end{pmatrix}$

$\Rightarrow \overrightarrow{AB}.\boldsymbol{n} = 0$

$\Rightarrow \begin{pmatrix}1\\3\\-1\end{pmatrix}.\begin{pmatrix}p\\q\\r\end{pmatrix} = 0;$ $\quad p + 3q - r = 0$………….(1)

$\Rightarrow \overrightarrow{AC}.\boldsymbol{n} = 0$

$\Rightarrow \begin{pmatrix}-4\\4\\1\end{pmatrix}.\begin{pmatrix}p\\q\\r\end{pmatrix} = 0; \quad -4p + 4q + r = 0$……………..(2)

(1)+(2)

$\Rightarrow -3p + 7q = 0; \quad p = \frac{7}{3}q$

putting$p$ in (1) gives $\frac{7}{3}q + 3q - r = 0; \quad \frac{7q+9q-3r}{3} = 0;\ 3r = 16q;\ \ r = \frac{16}{3}q$

$\Rightarrow p: q: r = \frac{7}{3}q: q: \frac{16}{3}q$

$\Rightarrow p: q: r = 7: 3: 16$

$\therefore \boldsymbol{n} = \begin{pmatrix}7\\3\\16\end{pmatrix}$

Using $\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}7\\3\\16\end{pmatrix} = \begin{pmatrix}1\\4\\6\end{pmatrix}.\begin{pmatrix}7\\3\\16\end{pmatrix}$

$\Rightarrow 7x + 3y + 16z = 7 + 13 + 96$

$\therefore 7x + 3y + 16z = 115$

Example 37

Find the equation of the plane which passes though the point P (2, 5,3) and containing the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+1}{1}$.

Solution

P (2, 5, 3), $b = 2i + 3j + k$, A(1, 2, -1)

$\overrightarrow{AP} = \begin{pmatrix}2\\5\\3\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\3\\4\end{pmatrix}$

Let $\boldsymbol{n} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$

$\Rightarrow \overrightarrow{AP}.\boldsymbol{n} = 0$

$\Rightarrow \begin{pmatrix}1\\3\\4\end{pmatrix}.\begin{pmatrix}a\\b\\c\end{pmatrix} = 0; \quad a + 3b + 4c = 0$………………(1)

$\Rightarrow \boldsymbol{b}.\boldsymbol{n} = 0$

$\Rightarrow \begin{pmatrix}2\\3\\1\end{pmatrix}.\begin{pmatrix}a\\b\\c\end{pmatrix} = 0; \quad 2a + 3b + c = 0$………………(2)

$(1) - (2)$

$\Rightarrow -a + 3c = 0; \ \ a = 3c$

Putting a in (1)

$\Rightarrow 3c + 3b + 4c = 0; \ \ b = -\frac{7}{3}c$

$\Rightarrow a: b: c = 3c: -\frac{7}{3}c: c$

$\Rightarrow a: b: c = 9: -7: 3$

$\Rightarrow \boldsymbol{n} = 9\boldsymbol{i} - 7\boldsymbol{j} + 3\boldsymbol{k}$

Using $\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 9 \\ -7 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ -7 \\ 3 \end{pmatrix}.\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

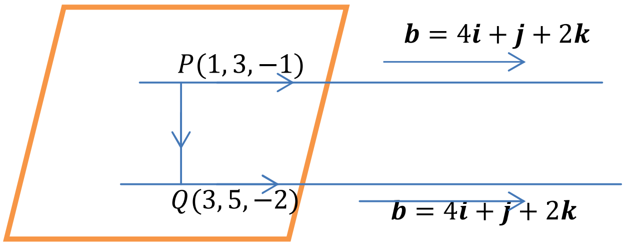
$\Rightarrow 9x - 7y + 3z = 9 - 14 - 3$

$\therefore 9x - 7x + 3z = -8$

Example 38

Find the equation of the plane which contains the parallel lines $\frac{x-1}{4} = \frac{y-3}{1} = \frac{z+1}{2}$ and $\frac{x-3}{4} = \frac{y-5}{1} = \frac{z+2}{2}$

Solution



$$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$\boldsymbol{n} = \overrightarrow{PQ} \times \boldsymbol{b}$

$$= \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & -1 \\ 4 & 1 & 2 \end{vmatrix}$$

$= 5\boldsymbol{i} - 8\boldsymbol{j} - 6\boldsymbol{k}$

Using $\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{p}.\boldsymbol{n}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} 5 \\ -8 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}.\begin{pmatrix} 5 \\ -8 \\ -6 \end{pmatrix}$$

$\Rightarrow 5x - 8y - 6z = 5 - 24 + 6$

$\therefore 5x - 8y - 6z = -13$

Example 40

Find the equation of the plane containing the two lines$\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$ and

$$\boldsymbol{r} = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

Solution

The two vectors in the plane are the directions of the two lines which are $\boldsymbol{b_1} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$and $\boldsymbol{b_2} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$.

Let $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the vector perpendicular to the plane.

$\Rightarrow \boldsymbol{b_1}.\boldsymbol{n} = 0$

$\Rightarrow \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0;\ -a + 3b - 4c = 0$…………(1)

$\Rightarrow \boldsymbol{b_1}.\boldsymbol{n} = 0$

$\Rightarrow \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0;\ 2a - b + 5c = 0$…………(2)

$2 \times (1) + (2)$gives

$5b - 3c = 0; b = \frac{3}{5}c$

Putting b into (1) we get

$-a + 3\left(\frac{3}{5}\right)c - 4c = 0$

$\Rightarrow -5a + 9c - 20c = 0;\ -5a - 11c = 0; a = -\frac{11}{5}c$

$\Rightarrow a:b:c = -\frac{11}{5}c:\frac{3}{5}c:c$

$\Rightarrow a:b:c = -11:3:5$

$\boldsymbol{n} = \begin{pmatrix} -11 \\ 3 \\ 5 \end{pmatrix}$

Using $\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} -11 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.\begin{pmatrix} -11 \\ 3 \\ 5 \end{pmatrix}$

$\Rightarrow -11x + 3y + 5z = (-11 \times 3) + (1 \times 3) + (2 \times 5)$

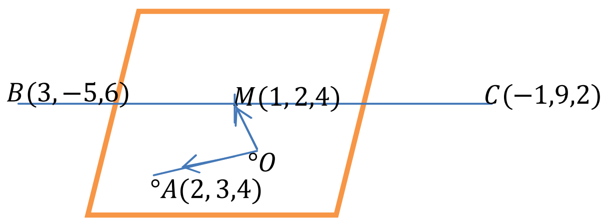
$\Rightarrow -11x + 3y + 5z = -20$

$\therefore$the equation of the plane is $11x - 3y - 5z = 20$

Example 41

Find the equation of the plane containing the point A (2, 3, 4) and equidistant from the points B (3, -5, 6) and C (-1, 9, 2).

Solution



When a plane is equidistant from two points then the mid-point M of $\overline{BC}$ lies on the plane

So let $\boldsymbol{n} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$be the vector perpendicular to both $\overrightarrow{OM}$ and $\overrightarrow{OA}$

$\Rightarrow \overrightarrow{OM}.\boldsymbol{n} = 0$

$\Rightarrow \begin{pmatrix}1\\2\\4\end{pmatrix}.\begin{pmatrix}a\\b\\c\end{pmatrix} = 0; a + 2b + 4c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$

$\Rightarrow \overrightarrow{OA}.\boldsymbol{n} = 0$

$\Rightarrow \begin{pmatrix}2\\3\\4\end{pmatrix}.\begin{pmatrix}a\\b\\c\end{pmatrix} = 0; 2a + 3b + 4c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$

(1)-(2)

$\Rightarrow -a - b = 0; a = -b$

Putting a into (1)

$\Rightarrow -b + 2b + 4c = 0; c = -\frac{1}{4}b$

$\Rightarrow a: b: c = -b: b: -\frac{1}{4}b$

$\Rightarrow a: b: c = -4: 4 - 1$

$\Rightarrow \boldsymbol{n} = -4\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$

Using $\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}-4\\4\\-1\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix}.\begin{pmatrix}-4\\4\\-1\end{pmatrix}$

$\Rightarrow -4x + 4y - z = (2 \times -4) + (3 \times 4) + (4 \times -1)$

$\Rightarrow -4x + 4y - z = 0$

∴the equation of the plane is $4x - 4y + z = 0$

Intersection of plane and a line

- We substitute the parametric equations of the line into the equation of the plane.

Example 42

Find the point of intersection of the line $\frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$ and the plane

$r.\begin{pmatrix}-1\\2\\3\end{pmatrix} = -1$

Solution

Let $\frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1} = t$

$\Rightarrow x = 1 + 2t, y = -t \text{ and } z = -3 + t$

Putting x, y and z into the plane

$\Rightarrow \begin{pmatrix}1+2t\\-t\\-3+t\end{pmatrix}.\begin{pmatrix}-1\\2\\3\end{pmatrix} = -1$

$\Rightarrow -1(1 + 2t) + 2(-t) + 3(-3 + t) = -1$

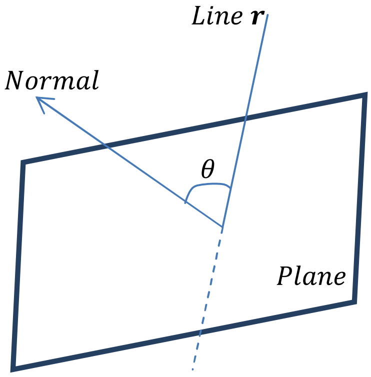
$\Rightarrow -1 - 2t - 2t - 9 + 3t = -1$

$\Rightarrow -t = 9 \therefore t = -9$

$$\Rightarrow \begin{pmatrix} 1+2t \\ -t \\ -3+t \end{pmatrix} = \begin{pmatrix} 1+2(-9) \\ -(-9) \\ -3-9 \end{pmatrix} = \begin{pmatrix} -17 \\ 9 \\ -12 \end{pmatrix}$$

$\therefore$the point of intersection of the line and the plane is $(-17, 9, -12)$

Angle between a line and a plane

- 

- If n and b is the normal to the plane and the directional vector to the line respectively, θ as the required angle then we can find it using the formula $\boldsymbol{n}.\boldsymbol{b} = |\boldsymbol{n}||\boldsymbol{b}|\cos(90^0 - \theta)$.

Example 43

Find the angle between the plane $3x + 4y - 5z = 6$ and the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$

Solution

Let $n = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ and θ be the required angle

Using $\boldsymbol{n}.\boldsymbol{b} = |\boldsymbol{n}||\boldsymbol{b}|\cos(90^0 - \theta)$.

$$\Rightarrow \cos(90^0 - \theta) = \frac{\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}.\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}}{\sqrt{3^2+4^2+(-5)^2}.\sqrt{1^2+5^2+(-3)^2}}$$

$$\Rightarrow \cos(90^0 - \theta) = \frac{(3\times1)+(4\times5)+(-5\times-3)}{\sqrt{50}.\sqrt{35}}$$

$$\Rightarrow \cos(90^0 - \theta) = \frac{38}{\sqrt{50\times35}}$$

$$\Rightarrow 90^0 - \theta = 24.72^0$$

$$\therefore \theta = 65.28^0$$

Angle between a plane and a plane

- Given that $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$ are the normal vector any given two plane and θ the angle between the planes, we use the formula $\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|\cos\theta$

Example 44

Find the angle between the planes $3x + 4y + 5z = 7$ and $x + 2y - 2z = 11$

Solution

Let $\boldsymbol{n_1} = \begin{pmatrix}3\\4\\5\end{pmatrix}, \boldsymbol{n_2} = \begin{pmatrix}1\\2\\-2\end{pmatrix}$ and θ be the required angle.

Using $\boldsymbol{n_1}.\boldsymbol{n_2} = |\boldsymbol{n_1}||\boldsymbol{n_2}|\cos\theta$

$\Longrightarrow \cos\theta = \frac{\begin{pmatrix}3\\4\\5\end{pmatrix}.\begin{pmatrix}1\\2\\-2\end{pmatrix}}{\sqrt{3^2+4^2+5^2}.\sqrt{1^2+2^2+(-2)^2}}$

$\Longrightarrow \cos\theta = \frac{3+8-10}{\sqrt{50}.\sqrt{9}}$

$\Longrightarrow \cos\theta = \frac{1}{3\sqrt{50}}$

$\therefore \theta = 87.3^0$

Intersection of planes

- Two planes intersect along a line and the equation of the line of intersection I obtained as follows:
- Three planes intersect in a point, we just solve the three equations simultaneously to get this point of intersection

Example 45

Find in both vector and Cartesian form the equation of the line of intersection of the planes $3x - 5y + z = 8$ and $2x - 3y + z = 3$

Solution

Eliminating z from the two equations by subtraction

$3x - 5y + z = 8$ ……………………….(1)

$2x - 3y + z = 3$ ………………………(2)

(1)-(2)

$\Longrightarrow x - 2y = 5$

Let $y = t$

$\Longrightarrow x = 5 + 2t$

Putting x and y into (1) we get

$3(5 + 2t) - 5(t) + z = 8$

$\Longrightarrow z = -7 - t$

$\Longrightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5+2t\\t\\-7-t\end{pmatrix}$

$\Longrightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\0\\-7\end{pmatrix} + t\begin{pmatrix}2\\1\\-1\end{pmatrix}$

$\therefore$the vector equation of the line of intersection is $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}5\\0\\-7\end{pmatrix} + t\begin{pmatrix}2\\1\\-1\end{pmatrix}$

$\therefore$the Cartesian equation of the line of intersection is $\frac{x-5}{2} = \frac{y}{1} = \frac{z+7}{-1}$

Method 2

The vectors $\begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ are perpendicular to the planes 1 and 2 respectively.

$\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & -\boldsymbol{j} & \boldsymbol{k} \\ 3 & -5 & 1 \\ 2 & -3 & 1 \end{vmatrix}$

$= -2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$is the directional vector to the line f intersection

Given the two equations of the plane

$3x - 5y + z = 8$

$2x - 3y + z = 3$

Let $y = 0$

$3x + z = 8$ ……………………….(1)

$2x + z = 3$ ……………………….(2)

(1)-(2)

$\Rightarrow x = 5\,; z = 8 - 3(5) = -7$

Then the point on the line is A (5, 0, -7)

The vector equation of the line is given by $r = \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

Which is the same as $\boldsymbol{r} = \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ where$t = -\lambda$

The Cartesian equation is given by $\frac{x-5}{2} = \frac{y}{1} = \frac{z+7}{-1}$

Example 46

Find the point of intersection of the three planes;

$2x + 3y + 4z = 8,\ \ 3x - 2y - 3z = -2,\ 5x + 4y + 2z = 3$

Solution

Using elimination method

$2x + 3y + 4z = 8$ …………………….(1)

$3x - 2y - 3z = -2$ …………………..(2)

$5x + 4y + 2z = 3$ …………………….(3)

$y$and$z$ may be eliminated easily, we eliminate y as follows

$2 \times (2) + (3)\!: 11x - 4z = -1$……………….(4)

$3 \times (3) - 4(1)\!: 7x - 10 = -23$……………….(5)

$5 \times (4) - 2 \times (5)\!:\ 41x = 41$

$x = 1, z = 3, y = -2$

$\therefore$ *the point of intersection is* $(1, 3, -2)$

Shortest distance of a point from a plane

- Give the plane $ax + by + cz + d = 0$ and the point $P(x_1, y_1, z_1)$

- The displacement of P from the plane is given by $\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$
- The distance is given by $\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$
- The distance of the plane from the origin we Put P(0, 0, 0) into the above formula we obtain, $distance = \left|\frac{d}{\sqrt{a^2+b^2+c^2}}\right|$

Example 47

Find the distance of the plane $2x - 4y + z = 7$ from the;

(a) point (1, 4, -3)

(b) origin

Solution

(a)

$\Rightarrow (1, 4, -3) \Leftrightarrow (x_1, y_1, z_1)and2x - 4y + z - 7 = 0 \Leftrightarrow\ ax + by + cz + d = 0$

Distance $= \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$

$= \left|\frac{(2\times1)+(-4\times4)+(1\times-3)-7}{\sqrt{2^2+(-4)^2+1^2}}\right|$

$= \left|\frac{2-16-3-7}{\sqrt{21}}\right|$

$= 5.2372$units

(b) using $distance = \left|\frac{d}{\sqrt{a^2+b^2+c^2}}\right|$

$= \left|\frac{-7}{\sqrt{2^2+(-4)^2+1^2}}\right|$

$= \left|\frac{-7}{\sqrt{21}}\right|$

$= 1.5275\ units$

Example 48

A plane contains a point A, position vector $3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$and is perpendicular to the vector $\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$. Find;

(a) Scalar product vector equation of the plane.

(b) The perpendicular distance of the plane from the origin.

(c) The perpendicular distance from this plane to the parallel plane $\boldsymbol{r}.(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) = -3$.

Solution

(a) Using $\boldsymbol{r}.\boldsymbol{n} = \boldsymbol{a}.\boldsymbol{n}$

$\boldsymbol{r}.(\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}) = (3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}).(\boldsymbol{i} + +2\boldsymbol{j} - 2\boldsymbol{k})$

$\boldsymbol{r}.(\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}) = 7$

(b) Using $distance = \left|\frac{d}{\sqrt{a^2+b^2+c^2}}\right|$

$= \left|\frac{7}{\sqrt{1^2+2^2+(-2)^2}}\right| = \left|\frac{7}{\sqrt{9}}\right| = \frac{7}{3}\ units$

(c) The plane $\boldsymbol{r}.(\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) = -3$ is on the other side of the origin and its perpendicular distance from the origin is $\frac{3}{\sqrt{9}} = 1\ unit$.

Thus the perpendicular distance between the planes is $\frac{7}{3}+1 = \mathbf{3\frac{1}{3}}\ \boldsymbol{units}$

Distance of a perpendicular from a point to a plane

- This involves obtaining the coordinates of the foot, distance and equation of the perpendicular from the point to the plane.

Example 49

Find the distance of the point A (25, 5, 7) from the plane$12x + 4y + 3z = 3$

Solution

Let P be the point such that $\overrightarrow{AP}$ is perpendicular to the plane.

Then the distance required is the length of vector $\overrightarrow{AP}$.

We know that $12\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$is a perpendicular vector to the plane.

Let $\overrightarrow{AP} = t(12\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$; then $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$\Rightarrow \overrightarrow{OP} = (25\boldsymbol{i} + 5\boldsymbol{j} + 7\boldsymbol{k}) + t(12\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$

$= (25 + 12t)\boldsymbol{i} + (5 + 4t)\boldsymbol{j} + (7 + 3k)\boldsymbol{k}$

Hence P is the point $(25 + 12t, 5 + 4t, 7 + 3t)$

Since the point P lies in the plane, its coordinates satisfy the equation of the plane; consequently

$\Rightarrow 12(25 + 12t) + 4(5 + 4t) + 3(7 + 3t) = 3$

$t = -2$

Hence $\overrightarrow{AP} = -2(12\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$

$= -24\boldsymbol{i} - 8\boldsymbol{j} - 6\boldsymbol{k}$

$|\overrightarrow{AP}| = \sqrt{(-24)^2 + (-8)^2 + (-6)^2}$

$= \sqrt{676}$

$= 26\ units$

∴the distance from the point A to the plane is 26 units.

Exercise 6.3

1. Find the Cartesian equation of the plane with vector equation $\boldsymbol{r}.(2i + 3j - 4k) = 5$.
2. A plane contains a point A, position vector $3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$and is perpendicular to the vector $\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$. Find the Cartesian equation of the plane.
3. Find the equation of the plane which passes through the point (4, 3, -2) and is perpendicular to the vector $\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$.
4. Find the equation of the plane which passes through the points

(a) A (1, 0, -1), B (3, 3, 2) and C (4, 5, -1)

(b) (1, 1, 0), (0, 1, 2) and (2, 3, -8)

5. Find the vector equation for the plane containing the three points A, B and C whose position vectors are $2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $5\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$

6. Find the equation of the plane which passes through the point (1, 2, 3) and which is parallel to h vectors $\begin{pmatrix} 2 \\ 4 \\ -10 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix}$
7. Find the equation of the line which is perpendicular to the plane $x + 2y - 3z = 10$and passes through the point (3, 4, 2)
8. Write down the equation of the plane parallel to the plane $\boldsymbol{r}.\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 6$ and passing through the point with position vector $4\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$
9. Given that A, B and C are points (1, 1, 1), (5, 0, 0) and (3, 2, 1) respectively, find the equation which must be satisfied by the coordinates (x, y, z) of any point P in the plane ABC. $Ans(x - 2y + 6z = 5)$
10. Given the lines $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$. If the two lines pass through the point A (1, 2, 3). Find the equation of the plane containing the two lines.
11. Find the equation of the plane containing the two lines $\boldsymbol{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$and $\boldsymbol{r} = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$. $Ans(11x - 3y - 5z = 20)$
12. Find the equation of the plane through (3, 2, 2) which is perpendicular to the line $\frac{x-1}{4} = \frac{y+2}{1} = \frac{z+3}{-3}$. $Ans(4x + y - 3z = 8)$
13. Find the equation of the plane containing the points A (2, 3, 4) and B (-1, 5, 6)
14. The planes P1 and P2 have Cartesian equations $2x + 5y - 14z = 30, 2x + 5y - 14z = -15$respectively. State the vector equation of P1 and P2 in scalar product form.
15. Find the position vector of the point of intersection of the line $\boldsymbol{r} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$and the plane $\boldsymbol{r}.\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 12$
16. Find the point where the line $\frac{x-3}{-1} = \frac{y-1}{2} = \frac{z+3}{4}$ cuts the plane $3x - y + 2z = 8$$Ans(1, 5, 5)$
17. Point $P(14 + 2t, 5 + 2t, 2 - t)$ lies on a fixed line for all values of t. Find the Cartesian equation of the line and find the cosine of the acute angle between the line and the plane $x = z$. $Ans\left(\frac{x-14}{2} = \frac{y-5}{2} = 2 - z, \cos\theta = \frac{\sqrt{2}}{2}\right)$
18. Find the equation of the common line (line of intersection) of the two planes $\pi_1: 3 - y - 5z = 7$and$\pi_2: 2x + 3y - 4z = -2$
$Ans\left(\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 19 \\ 2 \\ 11 \end{pmatrix}\right)$

19. Two planes $\boldsymbol{r}.\begin{pmatrix}2\\1\\3\end{pmatrix} = 11$ and $r.\begin{pmatrix}-1\\0\\1\end{pmatrix} = 6$ meet at a point whose position vector is $5\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$. Find the equation of the line of intersection of the two planes.

20. Find the acute angle between the line $\frac{x-2}{2} = \frac{y-1}{3} = \frac{z-3}{1}$ and the plane

$$2x + 4y + z = 9$$

21. Find the angle between the line $\frac{x+1}{4} = y - 2 = \frac{z-3}{-1}$ and the plane $3x - 5y + 4z = 5$ giving your answer to the nearest degree. $Ans(6^0)$

22. Find the angle between the planes $2x - y + z = 5$ and $x + y - z = 0$

23. Find the angle between the following planes $\boldsymbol{r}.(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = 7and\boldsymbol{r}.(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) = 10$

24. Find in both Cartesian and vector equation of the line of intersection of the two planes $7x - 4y + 3z = -3$ and $4x + 2y + z = 4$

$$Ans\left(\left(\frac{x-0}{1} = \frac{y-\frac{3}{2}}{-\frac{1}{2}} = \frac{z-1}{-3}\right), \left(r = \begin{pmatrix}0\\\frac{3}{2}\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-\frac{1}{2}\\-3\end{pmatrix}\right)\right)$$

25. Find the Cartesian equation of the line of intersection of the planes $2x - 3y + 2z = 5$and$3x - y + 4z = 5$

26. Find the distance between the parallel planes $2x + 2y - z = -4$and $6x + 4y - 2z = 6$. $Ans\left(\frac{14}{\sqrt{56}}\right)$units

27. With respect to the origin O, the points P, Q, R and S have position vectors given by $\boldsymbol{OP} = \boldsymbol{i} - \boldsymbol{k}, \boldsymbol{OQ} = -2\boldsymbol{i} + 4\boldsymbol{j}, \boldsymbol{OR} = 4\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{OS} = 3\boldsymbol{i} + 5\boldsymbol{j} - 6\boldsymbol{k}$
    - (i) Find the equation of the plane containing P, Q and R giving your answer in the form $ax + by + cz = d$. $Ans(2x + 3y - 6z = 8)$
    - (ii) The point N is the foot of the perpendicular from S to this plane. Find the position vector of N and show that the length of $\overrightarrow{SN}$ is 7.

28. Find the coordinates of the point N where the perpendicular from (37, 9, 10) meets the plane $12x + 4y + 3z = 3$ and also find;

(a) the equation of this perpendicular

(b) the perpendicular distance of the perpendicular.

29. Find the point of intersection of the following planes $2x + 3y + z = 8, x + y + z = 10, 3x + 5y + z = 6$

30. Find the distance to the plane $3x + 4y - 5z = 21$ from the origin. $Ans\left(\frac{21}{5\sqrt{2}}\right)$

31. Find the distance from the point (3, -2, 6) to the plane $3x + 4y - 5z = 21$. $Ans(5\sqrt{2})$

32. Find where the line from A(2, 7, 4) perpendicular to the plane $\pi: 3x - 5y + 2z + 2 = 0$, meets π. $Ans\left(3\frac{1}{2}, 4\frac{1}{2}, 5\right)$

33. The point A has position vector $2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$and the line l has equation $\boldsymbol{r} = -5\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k} + \lambda(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$

(i) Find the position vector of the point N on l such that $\overrightarrow{AN}$ is perpendicular to l.

(ii) Show that the perpendicular distance from A to l is $\sqrt{26}$.

The points B and C have position vectors $-5\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{k}$ and $6\boldsymbol{i} + 13\boldsymbol{j} - 7\boldsymbol{k}$ respectively and the point D is the mid-point of $\overrightarrow{BN}$.

(iii) Show that the plane ANC is perpendicular to l.

(iv) Find the acute angle between the planes ANC and ACD.

CHAPTER 7

7. COORDINATE GEOMETRY I

7.1 Straight Lines

- We are to look at lines of the form $y = mx + c$, where m is the gradient of the line and c the y-intercept.

Cartesian equation

- The equation of the form $y = mx + c$ is the Cartesian equation of the line

Gradient of a straight line

- When given two points $\mathbf{A}(\mathbf{x_1}, \mathbf{y_1})$ and $\mathbf{B}(\mathbf{x_2}, \mathbf{y_2})$ the gradient is given by

  $\mathbf{m} = \frac{\textbf{change in y}}{\textbf{change in x}} = \frac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}}$

  Example 1

  Find the gradient of the line passing through the points (4, 8) and (3, -2)

  Solution

  $m = \frac{-2-8}{3-4} = 10$

- Also gradient is given by the tangent of the angle that the line makes with the positive x-axis

  Example 2

  Find the gradient of the line that makes an angle of 73^0 with the positive x-axis.

  Solution

  $Gradient = \tan 73^0 = 3.2709$

- All parallel lines have the same gradient
- *And when two lines are perpendicular the product of their gradient is -1*

  $\boldsymbol{m_1 m_2 = -1}$

Intercepts

- The point where the line cuts the y-axis is called a y-intercept and at this point $x = 0$. It has coordinates of $(0, c)$
- The point where the line cuts the x-axis is called the x-intercept and at this point $y = 0$. It has coordinates of $(x_1, 0)$

Example 3

Find the x and y-intercepts of the line $2x - 3y = 6$.

Solution

At the $x - intercept, y = 0$

$\Longrightarrow 2x = 6; x = 3$

$\therefore$the x-intercept is $x = 3$

At the $y - intercept, x = 0$

$\Longrightarrow -3y = 6; y = -2$

$\therefore$the$y$ $-$intercept is $y = -2$

- We can find the points on a curve by substituting various values

Example 4

Find the y-coordinates of the points on the curve
$y = 2x^2 - x - 1$for which x=2, -3, 0.
Solution
$when\ x = 2, y = 2(2)^2 - (2) - 1 = 5$
$whenx = -3, y = 2(-3)^2 - (-3) - 1 = 20$
$when\ x = 0, y = 2(0)^2 - (0) - 1 = -1$

Example 5
Determine whether the following points lie on the following lines y=6x+7 (1, 13), y=2x+2 (13, 30)
Solution
For the point (1,13) and the equation $y = 6x + 7$
$when\ x = 1, y = 6(1) + 7 = 13$
$\therefore$the point (1, 13) lies on the line $y = 6x + 7$
For the point (13, 30) and the equation $\mathrm{y = 2x + 2}$
$when\ x = 13, y = 2(13) + 2 = 28$
$\therefore$the point (13, 30) doesn't lie on the line $\mathrm{y = 2x + 2}$

Mid-point of a line

- Given two points$\mathrm{A(x_1, y_1)}$and$\mathrm{B(x_2, y_2)}$the midpoint is obtained from

  $\mathrm{M} = \left(\frac{\mathrm{x_1+x_2}}{2}, \frac{\mathrm{y_1+y_2}}{2}\right)$
- The coordinates of the point P dividing the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $\lambda : \mu$ are $\left(\frac{\mu x_1 + \lambda x_2}{\lambda + \mu}, \frac{\mu y_1 + \lambda y_2}{\lambda + \mu}\right)$.
- The coordinates of the centre of gravity or centroid of the triangle formed by the points $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Example 6

Find the midpoint of a straight line joining the points A(6, 2) and B(8, 14)
Solution

The mid-point, $M = \left(\frac{6+8}{2}, \frac{2+14}{2}\right)$

$= (7,8)$

Example 7

The points P(4, -3), Q(-3, 4), R(-2, 7) and S are the vertices of a parallelogram PQRS. Find

(a) the coordinates of the mid-point of the diagonal PR.
(b) the coordinates of S.

Solution
(a) If M is the mid-point of the line joining P(4, -3) to R(-2, 7), M has coordinates $M\left(\frac{4+(-2)}{2}, \frac{(-3)+7}{2}\right) = M(1,2)$
$\therefore$Mid-point of PR is the point with coordinates (1, 2)
(b) Since the diagonals of a parallelogram bisect each other, $M(1,2)$ is also the mid-point of QS.
Suppose the coordinates of S are (x, y)
$\Rightarrow (1,2) = \left(\frac{-3+x}{2}, \frac{4+y}{2}\right)$
$\Rightarrow 1 = \frac{-3+x}{2} \Rightarrow \mathbf{x = 5}$
$\Rightarrow 2 = \frac{4+y}{2} \Rightarrow \mathbf{y = 0}$
$\therefore$The coordinates of S are (5, 0)

Qn. The points B and C have coordinates (1, 2) and (5, 3). If P and Q are the points which divide BC internally and externally in the ratio 1:2, find the coordinates of the mid-point of PQ. $Ans\left(-\frac{1}{3}, \frac{5}{3}\right)$.

Distance between two points

- *The distance between two points with given rectangular coordinates* $\boldsymbol{A}(\boldsymbol{x_1}, \boldsymbol{y_1})$ *and* $\boldsymbol{B}(\boldsymbol{x_2}, \boldsymbol{y_2})$ *is given by* $\boldsymbol{d} = \sqrt{\{(\boldsymbol{x_2} - \boldsymbol{x_1})^2 + (\boldsymbol{y_2} - \boldsymbol{y_1})^2\}}$

Example 8

Find the distance of a straight line joining the points A (3, 2) and B (8, 14)

Solution

$d = \sqrt{(8-3)^2 + (14-2)^2}$

$= \sqrt{5^2 + 12^2}$

$= \sqrt{169}$

$= 13\ units$

Example 9

Show that the triangle whose vertices are the points (-2, 2), (2, 3) and (-1, -2) is isosceles.

Solution

Let the coordinates be A (-2, 2), B (2, 3) and C (-1, -2)

$\Rightarrow |\overline{AB}| = \sqrt{(2 - -2)^2 + (3 - 2)^2}$

$= \sqrt{16 + 1}$

$= \sqrt{17} units$

$\Rightarrow |\overline{BC}| = \sqrt{(-1 - 2)^2 + (-2 - 3)^2}$

$= \sqrt{9 + 25}$

$= \sqrt{34} units$

$\Rightarrow |\overline{AC}| = \sqrt{(-1 - -2)^2 + (-2 - 2)^2}$

$= \sqrt{1 + 16}$

$= \sqrt{17} units$

$\therefore$Since $|\overline{AB}| = |\overline{AC}|$, the triangle is isosceles.

Equation of the line

- The equation of the line can be found in a number of ways

Example 10

Find the equation of the line with gradient 3 which cuts the y-axis at (0, 1)
Solution

$with\ m = 3, c = 1$the equation becomes $y = 3x + 1$.

Example 11

Find the equation of the straight line passing through (-5, 2) and (3, -4)

Solution

$\Longrightarrow m = \frac{-4-2}{3--5} = -\frac{6}{8} = -\frac{3}{4}$

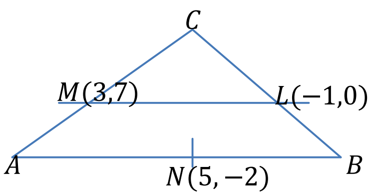
Using $y = mx + c$ and the point (-5, 2)

$\Longrightarrow 2 = \left(-\frac{3}{4}\right)(-5) + c; c = -\frac{7}{4}$

∴the equation is given by $y = -\frac{3}{4}x - \frac{7}{4}$

Example 12

L(-1, 0), M(3, 7) N(5, -2) are the midpoints of the sides BC, CA, AB respectively of triangle ABC find the equation of the line AB.
Solution



Gradient of ML $= \frac{0-7}{-1-3} = \frac{7}{4}$
Since line ML is parallel to line AB, then they have the same gradient and line AB contains the point N.
Using $m = \frac{7}{4}$ and $N(5, -2)$
$\Longrightarrow -2 = \left(\frac{7}{4}\right) \times 5 + c; c = -\frac{43}{4}$
$\Longrightarrow y = \frac{7}{4}x - \frac{43}{4}$
∴the equation of the line is given by $4y - 7x + 43 = 0$
Example 13

Find the equation the perpendicular bisector of the line joinning the points A(2, -3) and B(6, 5)

Solution
Gradient of line AB $= \frac{5--3}{6-2} = 2$
Gradient of the perpependiular bisector $= -\frac{1}{Grdadient\ of\ AB} = -\frac{1}{2}$
The perpendicular bisector passes through the mid-point M of line AB
$\Longrightarrow M = \left(\frac{6+2}{2}, \frac{5+(-3)}{2}\right) = (4,1)$
The equation of the perpendicular bisector can be obtained from

$1 = -\frac{1}{2}(4) + c; c = 3$

$\therefore$ the equation of the perpendicular bisector is $y = -\frac{1}{2}x + 3$ or $2y + x - 6 = 0$

Points of intersection

- To find the point where two lines meet we solve the two equations simultaneously.

Example 14

Find the equation of a line with gradient 3 and passing through the point of intersection of the two lines $2x - 3y = 6$ *and* $4x + y = 19$

Solution

$2x - 3y = 6$ ……………(1)

$4x + y = 19$ ……………..(2)

$2 \times (1) - (2)$to eliminate x

$\Rightarrow -7y = -7; y = 1$

From (1)

$\Rightarrow 2x - 3(1) = 6; x = \frac{9}{2}$

The point of intersection is $\left(\frac{9}{2}, 1\right)$

With $m = 3$ *and* $\left(\frac{9}{2}, 1\right)$

$\Rightarrow 1 = (3) \times \left(\frac{9}{2}\right) + c; \quad c = -\frac{25}{2}$

$y = 3x - \frac{25}{2}$

$\therefore$the equation of the line is $2y - 3x + 25 = 0$

Example 15

A straight line of the equation x-y=6, the line passed the curve $y^2 = 8x$ at P and Q. calculate the length of PQ

Solution

$\Rightarrow x = 6 + y$ ……………(1)

$y^2 = 8x$ ……………………(2)

Putting (1) into (2)

$\Rightarrow y^2 = 8(6 + y)$

$\Rightarrow y^2 - 8y - 48 = 0$

$\Rightarrow (y - 12)(y + 4) = 0$

$\Rightarrow y = 12, x = 6 + 12 = 18; \quad P(18, 12)$

$\Rightarrow y = -4, x = 6 - 4 = 2; \quad Q(2, -4)$

$\Rightarrow |\overline{PQ}| = \sqrt{(2 - 18)^2 + (-4 - 12)^2}$

$= \sqrt{512}$

$= 22.6274 units$

The area of the triangle whose vertices have given coordinates

- Consider the triangle ABC whose vertices are $\mathbf{A}(\mathbf{x_1}, \mathbf{y_1})$ $\mathbf{B}$ $(\mathbf{x_2}, \mathbf{y_2})$, $\mathbf{C}$ $(\mathbf{x_3}, \mathbf{y_3})$
- Area,$\Delta\mathbf{ABC} = \frac{1}{2}\{\mathbf{x_1}(\mathbf{y_2} - \mathbf{y_3}) + \mathbf{x_2}(\mathbf{y_3} - \mathbf{y_1}) + \mathbf{x_3}(\mathbf{y_1} - \mathbf{y_2})\}$… (i)

- If we take point C to be the origin 0 ($\mathbf{x_3} = \mathbf{y_3} = \mathbf{0}$)the points are$\mathbf{0}$ $(\mathbf{0}, \mathbf{0})$, $\mathbf{A}(\mathbf{x_1}, \mathbf{y_1})$, $\mathbf{B}$ $(\mathbf{x_2}, \mathbf{y_2})$
- The formula reduces to Area, $\Delta\mathbf{0AB} = \frac{\mathbf{1}}{\mathbf{2}}\{\mathbf{x_1y_2} - \mathbf{x_2y_1}\}$
- For the formula (i) to give a positive value for the area, it is necessary to take the points A, B and C in a special order.
- We can also find the area of a triangle or any quadrilateral by plotting it and removing the unnecessary areas.

Example 16

Find the area of the triangle formed by the points (-2, 3), (-7, 5), (3, -5)

Solution

$(-2, 3) \Leftrightarrow (x_1, y_1)$, $(-7,5) \Leftrightarrow (x_2, y_2)$ and $(3, -5) \Leftrightarrow (x_3, y_3)$

Using ,$\Delta = \frac{\mathbf{1}}{\mathbf{2}}\{\mathbf{x_1}(\mathbf{y_2} - \mathbf{y_3}) + \mathbf{x_2}(\mathbf{y_3} - \mathbf{y_1}) + \mathbf{x_3}(\mathbf{y_1} - \mathbf{y_2})\}$

$= \frac{1}{2}\{-2(5 - -5) + (-7)(-5 - 3) + 3(3 - 5)\}$

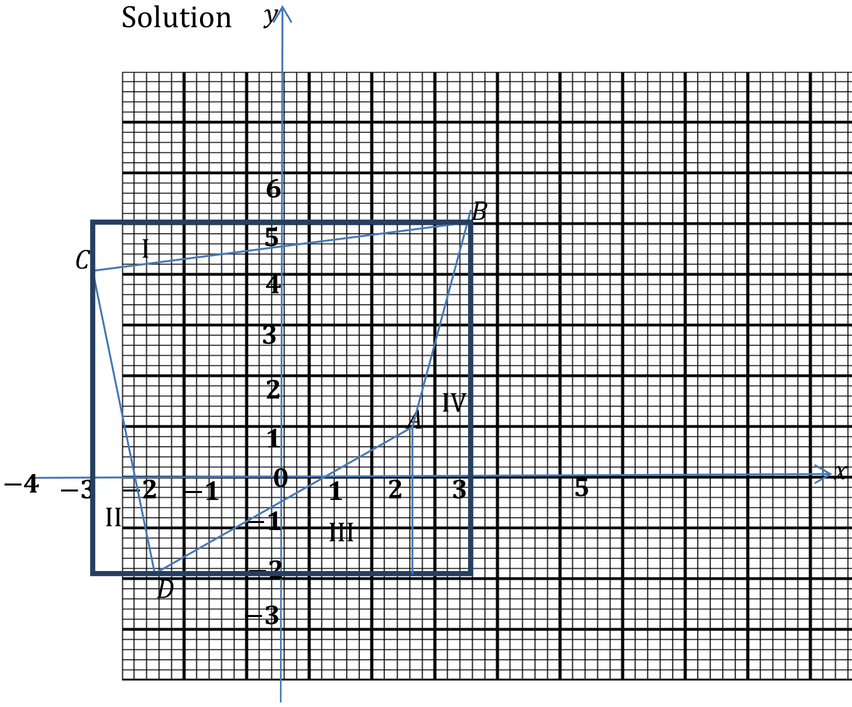
$= \frac{1}{2}\{-2(10) + (-7)(-8) + 3(-2)\}$

$= 15\ units^2$

Example 17

Find the area of the quadrilateral whose vertices are the points A (2, 1),B (3, 5), C (-3, 4) and D (-2, -2)

Solution



Here we use area subtraction method

Area of ABCD $= Area\ of\ rectange - Area(\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV})$

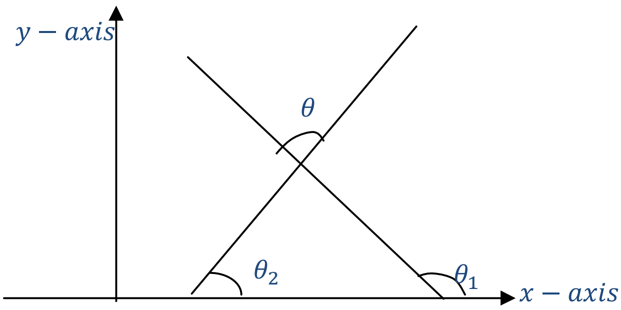
$= (7 \times 6) - \left\{\frac{1}{2}(6 \times 1) + \frac{1}{2}(6 \times 1) + \frac{1}{2}(4 \times 3) + \frac{1}{2}(1)(3 + 7)\right\}$

$= (42) - (3 + 3 + 6 + 5)$

$= 25\ units^2$

The angle between two straight lines

- Consider the two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ and suppose that they make angles of θ_1 and θ_2 respectively with the positive x-axis. Let θ be the acute angle between the two lines, then



$\Longrightarrow \theta_2 + \theta = \theta_1$ *(sum of 2 interior angles make up 1 exterior angle)*

$\Longrightarrow \theta = \theta_1 - \theta_2$

$\Longrightarrow \tan\theta = \tan(\theta_1 - \theta_2)$

$\Longrightarrow \tan\theta = \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1\tan\theta_2}$

But $\tan\theta_1$ and $\tan\theta_2$ are the gradients of these lines and hence $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

$\therefore \boldsymbol{\tan\theta = \frac{m_1 - m_2}{1+m_1m_2}} \Longrightarrow \theta = \tan^{-1}\left\{\frac{m_1 - m_2}{1+m_1m_2}\right\}$

- Note the following;

  (i) That the angle θbetween the lines depends only on the gradients of the lines, the constants $c_1$and $c_2$do not affect the angle.

  (ii) If the value of $\tan\theta$ is positive then it is the tangent of the acute angle between the two lines and if it is negative then it is the tangent of the obtuse angle between the two lines.

Examples 18

Find the angles between the following pairs of lines

(i) $y = 2x + 5$ *and* $3x + y = 7$

(ii) $3x - y + 7 = 0$ *and* $x - 3y + 8 = 0$

Solution

(i) $\Longrightarrow m_1 = 2, m_2 = -3$, let the angle be α

$\Longrightarrow \alpha = \tan^{-1}\left\{\frac{2--3}{1+(2)(-3)}\right\}$

$\therefore \alpha = 135^0$

(ii) $\Longrightarrow m_1 = 3, m_2 = \frac{1}{3}$, let the angle be β

$\Longrightarrow \beta = \tan^{-1}\left\{\frac{3-\frac{1}{3}}{1+(3)\left(\frac{1}{3}\right)}\right\}$

$\therefore \beta = 53.13^0$

Example 19

Find the tangent of the acute angle between the pairs of the lines whose equations are $3y = x - 7$ and $2y = 3 - 4x$

Solution

$\Longrightarrow m_1 = \frac{1}{3}, m_2 = -2$, let the angle be α

$\Longrightarrow \tan\alpha = \frac{\frac{1}{3}-(-2)}{1+\left(\frac{1}{3}\right)(-2)}$

$= 7$

∴the tangent of the acute angle is 7.

Perpendicular distance between a point and a line

- The perpendicular distance of the point $\mathbf{P}(\mathbf{x_1}, \mathbf{y_1})$ to the line $\mathrm{ax + by + c = 0}$ is given by $\left|\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}\right|$.
- The displacement of the point $\mathbf{P}(\mathbf{x_1}, \mathbf{y_1})$ from the line $\mathrm{ax + by + c = 0}$ is given by $\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}$.

Example 20

Find the distance of the points (2, -1) from the line $3x + 4y = 6$

Solution

Using

Distance $= \left|\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}\right|, a = 3, b = 4, c = -6, x_1 = 2, y_1 = -1$

$= \left|\frac{(3\times2)+(4\times-1)-6}{\sqrt{3^2+4^2}}\right|$

$= \left|\left(\frac{-4}{5}\right)\right|$

$= \frac{4}{5}\ units$

Example 21

Find the perpendicular distance from the line $3y = 4x - 1$ to the points

(a) (1, 3)

(b) (1, -2)

Solution

(a) Using

Distance $= \left|\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}\right|, -4x + 3y + 1 = 0;\ a = -4, b = 3, c = 1, x_1 = 1, y_1 = 3$

$= \left|\frac{(-4\times1)+(3\times3)+1}{\sqrt{(-4)^2+3^2}}\right|$

$= \left|\left(\frac{6}{5}\right)\right|$

$= \frac{6}{5}\ units$

(b) Distance $= \left|\frac{\mathrm{ax_1+by_1+c}}{\sqrt{\mathrm{a^2+b^2}}}\right|, -4\mathrm{x} + 3\mathrm{y} + 1 = 0;\ \mathrm{a} = -4, \mathrm{b} = 3, \mathrm{c} = 1, \mathrm{x_1} = 1, \mathrm{y_1} = -2$

$= \left|\frac{(-4\times1)+(3\times-2)+1}{\sqrt{(-4)^2+3^2}}\right|$

$= \left|\left(\frac{-9}{5}\right)\right|$

$= \frac{9}{5}\ units$

Note

$\left\{\left(\frac{-6}{5}\right),\ \left(\frac{9}{5}\right)\right\}$are the displacements of the points from the line

Since the two values are of opposite signs, this means that the two points are on opposite sides of the line.

Example 22

Find the equation of the straight line through P(7, 5) perpendicular to the line AB whose equation is $3x + 4y - 16 = 0$. Calculate the length of the perpendicular from P to AB.

Solution

Consider the line $3x + 4y - 16 = 0, m_1 = -\frac{3}{4}$

The gradient of the perpendicular, $m_2 = -1/\left(\frac{-3}{4}\right) = \frac{4}{3}$

The equation can be obtained from $y = mx + c;\quad P(7,5)$

$\Rightarrow 5 = \frac{4}{3}(7) + c; c = \frac{-13}{3}$

$\therefore y = \frac{4}{3}x - \frac{13}{3}$ $or\ 3y - 4x + 13 = 0$is the equation of the perpendicular.

Distance $= \left|\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}\right|$, $3x + 4y - 16 = 0;\ a = 3, b = 4, c = -16, x_1 = 7, y_1 = 5$

$= \left|\frac{(3\times7)+(4\times5)-16}{\sqrt{3^2+4^2}}\right|$

$= 5\ units$

Example 23

Find the equations of the lines that bisect the angle between the lines

$3x - 4y + 13 = 0$and$12x + 5y - 32 = 0$

Solution

Any point on a line that bisects one of the angles between the line $3x - 4y + 13 = 0$ and the line $12x + 5y - 32 = 0$ will be equidistant from these two lines.

Thus we require the locus of all the points which are equidistant from

$3x - 4y + 13 = 0$and$12x + 5y - 32 = 0$.

Suppose that (x, y) is the general point equidistant from the two lines, then using $\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}$it follows that $\frac{3x-4y+13}{\sqrt{3^2+4^2}} = \pm\frac{12x+5y-32}{\sqrt{12^2+5^2}}$

$\Rightarrow 13(3x - 4y + 13) = \pm5(12x + 5y - 32)$

$\Rightarrow 3x + 11y - 47 = 0$ and$11x - 3y + 1 = 0$

$\therefore$The equations of the two bisectors of the angle between the lines $3x - 4y + 13 = 0$ and $12x + 5y - 32 = 0$ are $\underline{3x + 11y - 47 = 0}$and $\underline{11x - 3y + 1 = 0.}$

Exercise 7.1

1. The points P(2, 3), Q(-11,8) and R(-4, 5) are vertices of a parallelogram PQRS which has PR as a diagonal. Find the coordinates of vertex S.
2. A triangle has vertices at A(2, 4), B(4, -2) and C(8, 12). L is the mid-point of AB and M is the mid-point of BC. Find;

(a) the coordinates of point L
(b) the coordinates of point M
(c) the distance LM.

3. Find the length between the points (4, -7) and (-1, 5) $Ans\ (13\ units)$
4. Find the length of the sides of a triangle whose vertices are the points (5, -6), (-3, -2) and (1, -3)
5. The points A, B and C have coordinates (2, 1), (7, 3) and (5, k) respectively. If AB and BC are of equal length, find the possible values of k. $Ans(k = 8\ and\ k = -2)$
6. Prove that the points A (-3, 4), B (1, 1) and C (7, 9) are the vertices of a right-angled triangle.
7. The points A, B and C have coordinates (-3, 2), (-1, -2) and (0, k) respectively, where k is a constant. Given that $|\overline{AC}| = 5|\overline{BC}|$, find the possible values of k. $Ans\left(k = -\frac{7}{3}, -2\right)$
8. A triangle has vertices (1, 2), (13, 7) and (6, 14). Prove that the triangle is isosceles.
9. Prove that the triangle with vertices P (2, 1), Q (5, -1) and R (9, 5) is a right-angled.
10. By first showing that the triangle with vertices A (-3, -1), B (1, -4) and C (7, 4) is right angled, deduce that the area of the triangle ABC is 25 units^2.
11. Find the coordinates of the mid-point M of the straight line joining A (4, -2) and B (-3, -1). Verify that the distance AM is equal to the distance BM.
12. Find the distance between the points A (1, 3) and B(6, 15) $Ans(13units)$
13. Point A and B have coordinates (1, 1) and (5, -1) respectively. Find, by calculation, which one of the points C (4, -3), D (10, 3) or E (5, -1) lies on the perpendicular bisector of AB.
14. A triangle has vertices at A(0,7), B(9, 4) and C(1, 0). Find;

(a) the equation of the perpendicular from C to AB.
(b) the equation of the straight line from A to the mid-point of BC.

15. Show that one of the angles of the triangle whose vertices are the points (5, 1), (-3, 7) and (8, 5) is a right angle.
16. A line is drawn through the point (2, 3) making an angles of 45^0 with the positive direction of the x-axis, and it meets the line x=6 at P. Find the distance of P from the origin O, and the equation of the line through P perpendicular to OP.

    $Ans(\sqrt{85}; 6x + 7y - 85 = 0)$
17. Find the point at which the line $3y - 12 = 4x$ cuts the y-axis, the x-axis and hence sketch the curve.
18. Find the length of the sides of the triangle formed by the three straight lines

$$x - 4y = 6, 3x - 2y + 1 = 0\ and\ x + y = 2$$

19. The line $y - 2x + 3 = 0$ intersect the curve $y = x^2 - 2x$ at the point A and B. Find the coordinates of A and B. $Ans\{(3,3)\ and\ (1,-1)\}$
20. Find the point at which the curve $y = 3x^2$and $y = x^2 - 5x - 3$ intersect. $Ans\left\{\left(\frac{-3}{2}, \frac{27}{4}\right), (-\mathbf{1}, \mathbf{3})\right\}$
21. The line $y = ax - 1$ and the curve $y = x^2 + bx - 5$ intersect at the points P(4, -5) and Q. Find the values of a and b and the coordinates of Q.
22. Find the point of intersection of the curve $(x-3)^2 + (y-4)^2 = 25$and the line $y + x = 1$
23. Prove, by calculation that the triangle formed by the lines $x - 2y + 1 = 0, 9x + 2y - 11 = 0$ and $7x + 6y - 53 = 0$ is isosceles. Calculate to the nearest degree, the smallest angle of the triangle.
24. Find the area of the triangle formed by the line $3x - 7y + 4 = 0$and the axes
25. Find the equation to the line which passes through the point (2, 3) and the point of intersection of the lines $2x + 3y - 1 = 0\ and\ 3x - 4y - 6 = 0$
26. Find the equation of a line which is parallel to the line $x + 4y - 1 = 0$ and which passes through the point of intersection of the lines $y = 2x$ and $x + y - 3 = 0$ $Ans(x + 4y - 9 = 0)$
27. Find the equation of the lines which pass through the point of intersection of the lines $x - 3y = 4$ and $3x + y = 2$, and are respectively parallel and perpendicular to the line $3x + 4y = 0$($\mathbf{3x + 4y + 1 = 0}$; $\mathbf{4x - 3y - 7 = 0}$)
28. Find the equation of the line joining the feet of the perpendicular s drawn from the point (1, 1) to the lines $3x - 3y - 4 = 0$ and $3x + y - 6 = 0$ $Ans(13x + y - 22 = 0)$
29. The points A(2, 1), $P(\alpha, \beta)$ and point B(1, 2) lie in the same plane. PA meets the x-axis at the point (h, 0) and PB meets the y-axis at the point(0, k). Find h and k in terms of $\alpha$and β.
30. Find the orthocentre (the point of intersection of the altitudes) of the triangle with vertices at A(-2, 1), B(3, -4) and C(-6, -1).
31. A line with a variable gradient m passes through the point (6, 2) and cuts the y-axis at P and the x-axis at Q,

    (i)Find the coordinates of P and Q in terms of m.
    (ii) If M is the mid-point of PQ, find the coordinates of M in terms of m.
32. The points A and B have coordinates (0, 1) and (4, 3) respectively;

    (i) Find the equation of the line passing through B and makes angle of $45^0$with AB.
    (ii) If the line in (i) above cuts the x-axis at C, find the coordinates of C.
    (iii) Given that the point A, B and C are joined to form a triangle ABC, calculate the area of the triangle ABC.
33. A line $L_1$with a positive gradient passes through the point (2, 0). It is also inclined at 45^0 to another line L_2. L_2 Passes through the point(1, 0) and makes an angle θ with the positive x-axis.

(a) Show that the gradient, m of L_2 is given by $\boldsymbol{m} = \frac{\mathbf{1+\tan\theta}}{\mathbf{1-\tan\theta}}$.

(b) Given that $\theta = \tan^{-1}\left[\frac{1}{3}\right]$, find the equation of a line L_3 which passes through the point (2, 0) and is perpendicular to line L_2

(c) Find the point of intersection of;

(i) Line L_1 and linen L_2

(ii) Line L_1 and line L_3

34. (a) Show that the mid-point of the line joining the fixed point P(7, 8) to the variable point $Q(2t+3, t-9)$lie on the line $x-2y=6$

35. The point A(1, 1) lies on the line PQ such that $|AP| = |AQ| = 2\sqrt{13}$. A perpendicular from point B(3, -2) meets PQ at A. Find the coordinates of P and Q

36. A line L has gradient and passes through the point (-1, 0). A line M passes through the points (0, 3) and (6, 0).

(a) Show that the line L and M are perpendicular.

(b) Find the coordinates of their point of intersection.

(c) Calculate the area enclosed by the lines L, M and the x-axis.

37. (a) Find the points of intersection of the line $y = x + 12$ and the curve $y = x^2 - 3x$.

(b) Given the points P(2, 8), Q(8, 6) and R(1, -1), find the;

(i) equation of the perpendicular bisector of PQ

(ii) point of intersection of the perpendicular bisector in (i) above and the line segment QR.

38. Find the equations of the lines that bisect the angles between the following pairs of lines

(a) $3x + y + 3 = 0, 3y = x - 1$($\boldsymbol{y = 2x + 1;\ x + 2y + 2 = 0}$)

(b) $y = x + 1\ , 7x + y + 3 = 0$($\boldsymbol{x + 3y = 1;\ y = 3x + 2}$)

(c) $3x = 4y + 2, 12y = 5x + 2$ $\quad Ans(7x + 4y = 18;\ 7y = 4x - 1)$

39. Find which two of the points A(1, 1), B(0, -1) and C(-1, 2) lie on the same side of the line $2x + 3y = 1$ and find how far the third is from the line.$Ans(A, C, \frac{4}{13}\sqrt{13}\ )$

40. Find the possible values of k given that the point (4, k) is the same distance from $9x + 8y + 1 = 0$ as (2, 5) is from $y = 12x + 2$. $\quad Ans(-2, 7\frac{1}{4})$

41. Find the locus of the points that are equidistant from the two lines $6y = 7x + 1, 9x + 2y + 3 = 0$.$Ans(x + 4y + 1 = 0, y = 4x + 1)$

Exercise 7.2

1. P, Q, R are the points (5, -3), (-6, 1), (1, 8) respectively. Show that the triangle PQR is isosceles, and find the coordinates of the mid-point of the base. $Ans(-5/2\ , 9/2)$
2. Three of the following four points lie on a circle whose centre is at the origin. Which are they, and what is the radius of the circle? $A(-1, 7), B(5, -5), C(-7, 5), D(7, -1)$. $Ans(A, B, D\ \sqrt{50})$
3. A and B are the points (12, 0) and (0, -5) respectively. Find the length of AB and the length of the median through the origin O, of the triangle OAB. $Ans\left(13,\ \ 6\frac{1}{2}\right)$
4. Show that the three give points are in each case collinear

(a) (-3, 1), (1, 2), (9, 4) (b) (0, 0), (3, 5), (21, 35)

5. Show that A (-3, -1), B (1, 2), C (0, -1), D (-4, -2) are the vertices of a parallelogram.
6. Show that P (1, 7), Q (7, 5), R (6, 2), S (0, 4) are vertices of a triangle. Calculate the lengths of the diagonals, and find their point of intersection $Ans\left(\sqrt{50};\ \left(3\frac{1}{2}, 4\frac{1}{2}\right)\right)$
7. Find the intercepts on the axes made by the straight line $3x - 2y + 10 = 0$. Hence find the area of the triangle enclosed by the axes and this line. $Ans(-10/3\,, +5;\ 25/3)$
8. Find the equation of the line joining the origin to the mid-point of the line joining A (3, 2) and B (5, -1). $Ans(y = \frac{1}{8}x)$
9. P (-2, -4), Q (-5, -2), R (2, 1), S are the vertices of a parallelogram. Find the coordinates of M, the point of intersection of the diagonals, and of S. $Ans\left(M\left(0, -\frac{3}{2}\right), S(5, -1)\right)$
10. Find the points of intersection of the following pairs of straight lines:

(a) $2x - 3y = 6\ and\ 4x + y = 19$

(b) $y = 3x + 2\ and\ 2x + 3y = 17$

11. P, Q, R are the points (3, 4), (7, -2), (-2, -1) respectively. Find the equation of the median through R of the triangle PQR. $Ans(2x - 7y - 3 = 0)$
12. Find the circumcentre of the triangle with vertices (-3, 0), (7, 0), (9, -6). Show that the point (1, 2) lies on the circumcircle. $Ans(2, -5)$
13. ABCD is a rhombus. A is the point (2, -1), and C is the point (4, 7). Find the equation of the diagonal BD. $Ans(x + 4y - 15 = 0)$
14. Find the equation of the line joining the points (6, 3) and (5, 8). Show also that these points are equidistant from the point (-2, 4). $Ans(5x + y - 33 = 0)$
15. Find the equation of the straight line, giving each in the form $ax + by + c = 0$:

(a) the line joining the points (2, 4) and (-3, 1)

(b) the line through (3, 2) parallel to the line $3x + 5y = 6$.

(c) The line through (3, -4) perpendicular to the line $5x - 2y = 3$

$$Ans\big((a)3x - 5y + 14 = 0, (b)3x + 5y - 14 = 0, (c)2x + 5y + 14 = 0\big)$$

16. Find the relation between x and y if the point (x, y) lies on the line joining the points (2, 3) and (5, 4). $Ans(x - 3y + 7 = 0)$
17. Find the coordinates of the points dividing the line joining the points (7, -5) to the point (-2, 7) internally in the ratio 5:4 and externally in the ratio 3:2.
18. If x and y are the coordinates of the middle point of the line joining the points (2, 3) and (3, 4), show that $x - y + 1 = 0$.
19. Show that the four points (2, 9), (-3, 12), (-8, 15) and (7, 6) all lie on the same straight line.
20. Fid the area of the quadrilateral whose angular points are (1, 1), (3, 5), (-2, 4) and (-1, -5).
21. Find the coordinates of the centre of the circumscribed circle of the triangle whose vertices are the points (-2, 2), (1, -2), (1, 3).
22. Find the area of the triangle whose vertices are the points (2, 1), (3, -2) and (-4, -1)
23. Find the area of the quadrilateral whose vertices are the points (1, 1), (2, 3), (3, 3) and (4, 1)
24. Find the coordinates of the middle point of the line joining the common points of the line $2x - 3y + 8 = 0$and $y^2 = 8x$.
25. Find the values of a and b if the straight line $ax + 5y = 7$ and $4x + by = 5$ intersect at the point (2, -1). If the lines meet the x-axis at A and B, find the length AB.

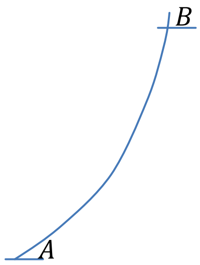
26. Show that the coordinates of the common point of the line $y = mx + a/m$ and the curve $y^2 = 4ax$ are $(a/m^2, 2a/m)$.
27. Show that the length of the line joining the common points of the line $y = mx + c$ and the curve $y^2 = 4x$ is $\frac{4}{m^2}(1+m^2)^{\frac{1}{2}}(1-mc)^{\frac{1}{2}}$
28. Find the equation to the straight line which passes through the point (1, 2) and makes an angle of 45^0 with the x-axis. $Ans(x - y + 1 = 0)$
29. Find the equation of the line which makes intercepts of -5 and 3 respectively on the axes of $x$and y.
30. Find the coordinates of the point of intersection P of the two straight lines $4x + 3y = 7, 3x - 4y = -1$. Find also the equation to the line joining P to the point (-2, 3)
31. Find the distance between the two parallel straight lines $2x + y = 4$ *and* $4x + 2y = 2$
32. Find the equation to the straight line passing through the point of intersection of the two lines $2x + 3y = 4$ and $3x - 2y = 5$ and also through the point of intersection of $3x - 4y = 7$ and $2x + 5y = 2$.
33. Find the equation to a straight line which passes through the point (3, 5) and makes equal intercepts on the coordinate axes.
34. Find the equations to the lines through the point (2, 3) which make angles of 45^0 with the line $x - 2y = 1$. $Ans(3x - y = 3, x + 3y = 11)$
35. Find the equations to the straight lines passing through the point (3, -2) and making angles of 60^0 with the line $\sqrt{3}x + y = 1$.
36. The base of an isosceles triangle lies along the line $3x + 2y = 2$ and one of the equal sides lies along $y = 2x$. Find the equation to the other equal side if it too passes through the origin.
37. A triangle ABC is formed by the lines $3x - 4y + 3 = 0\ (AB), x + y - 3 = 0\ (BC) and\ 4x - 3y - 5 = 0\ (AC)$. Find the equation to the straight line through C perpendicular to AB.
38. Show that the points (2, -1) and (1, 1) are on the opposite sides of the line $3x + 4y = 6$.
39. Find the equation to the lie which passes through the point (3, 2) and the point of intersection of the lines $2x + 3y - 1 = 0$and $3x - 4y - 6 = 0$.

    $Ans(43x - 29y = 71)$
40. Show that the point (1, 1) is equidistant from the lines $3x + 4y = 12, 5x - 12y + 20 = 0, 4x - 3y = 6$.
41. A and B are the points (3, 5) and (-5, -7) respectively. Find the co-ordinates of the points which divide AB internally and externally in the ratio 3:1
42. Find the coordinates of the mid-point of the line joining the points A (5 6) and B (11, 2).
43. P an Q are the points dividing the line joining A (-3, -4), B (5, 12) internally and externally in the ratio 5:3. Find the co-ordinates of P and Q.
44. The line joining the points A(3, 4) and B (7, 6) meets the line joining C (1, 3) and D (11, 8) at the point P. Given P is the mid-point of AB, find its co-ordinates and hence find the ratio CP:PD.
45. If the points A (5, 6, P (x, y) and B (2, 3) are collinear show that $x - y + 1 = 0$.

46. Show that the three lines $3x - 2y + 1 = 0, x + 2y + 3 = 0, 7x - 2y + 5 = 0$ pass through the same point.
47. Find the acute angle between the lines $y = \frac{1}{3}x + \frac{4}{3}; y = \frac{1}{2}x + \frac{5}{6}$. $Ans(8^0 8')$
48. Find the length of the perpendicular from the point P (2, -4) to the line $3x + 2y - 5 = 0$ and state which side of the line P is on.
49. Find the equation of the two lines through the point of intersection of the lines $3x + 2y - 1 = 0$ and $2x - y + 7 = 0$ which are also;
(a) Perpendicular to $3x + 2y - 1 = 0$
(b) Perpendicular to $2x - y + 7 = 0$
50. Find the equation of the line joining the points (3, 6), (5, 7) and show that it is perpendicular to the line joining the points (-3, 4), (-2, 2).

CHAPTER 8
DIFFERENTIATION 1
8.1 Gradient of a Curve

- We already know that the gradient of a straight line which passes through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$.
- We have also seen that the gradient of a straight line is the same all through, which is not true with a curve.
- Consider the curve below;

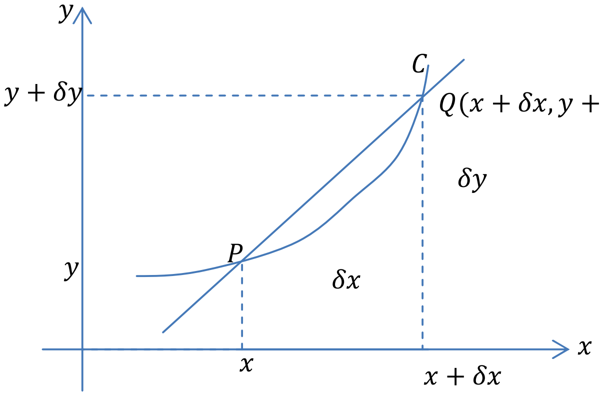


- The gradient changes as you move from A to B.
- So the **gradient of a curve** at a point P is the gradient of the **tangent line** to the curve at point P.

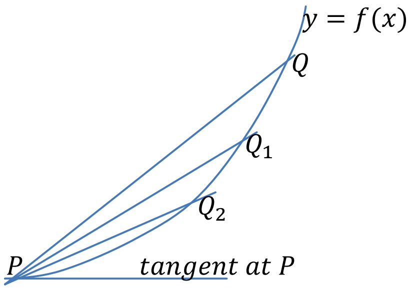
8.2 Gradient Function

Differentiation from first principles

- Consider the point $P(x_1, y_1)$ on the curve C, shown in the diagram below;



- We want the gradient of a curve at $P(x, y)$
- Let Q be a different point on the curve such that it has coordinates $(x + \delta x, y + \delta y)$ where δx and δy are small.
- The straight line PQ is called a chord of the curve C.
- As the distance δx becomes smaller, point Q moves closer to P and the chord PQ approaches the position of the tangent at P.



- The gradient of PQ is given by; $m_{PQ} = \frac{(y+\delta y)-y}{(x+\delta x)} = \frac{\delta y}{\delta x}$
- As δx tends to zero $(\delta x \to 0)$, $\frac{\delta y}{\delta x}$ approaches the value of the gradient of the tangent line at P. This value is called the limiting value of $\frac{\delta y}{\delta x}$ and is written as $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$
- The limiting value of $\frac{\delta y}{\delta x}$ is called the **differential coefficient or first derivative of y with respect to x** and is denoted by $\frac{dy}{dx}$.
- The process of finding this limiting value is called **differentiation**

Note:

- It is important to understand that $\frac{dy}{dx}$ does not mean dy divided by dx. It means the derivative of y with respect to x.
- The $\frac{d}{dx}$ is an operator, operating on the function y. To see this more clearly we can write $\frac{dy}{dx}$ as $\frac{d(y)}{dx}$.

Example 1

Given that $y = x^2$, find $\frac{dy}{dx}$ from first principles.

Solution

As x increases by δx, y will increase by δy

$\Rightarrow y + \delta y = (x + \delta x)^2$

$= x^2 + 2x\delta x + (\delta x)^2$

$\Rightarrow \delta y = x^2 + 2x\delta x + (\delta x)^2 - x^2$

$\Rightarrow \delta y = 2x\delta x + (\delta x)^2$

Dividing through by δx, gives

$\frac{\delta y}{\delta x} = 2x + \delta x$

As $\delta x \to 0, \frac{\delta y}{\delta x} \to \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = 2x$

Note:

The expression $\frac{dy}{dx} = 2x$ is called the **derived expression** or the **gradient function** or the **first derivative** of y with respect to x.

If the gradient of the curve $y = x^2$ at the point P(3, 9) is required. It is given by $\frac{dy}{dx} = 2(3) = 6$ meaning $\frac{dy}{dx}$ is evaluated when $x = 3$.

Alternative notation

- If the equation of a curve is denoted by $y = f(x)$ it is sometimes more convenient to denote $\frac{dy}{dx}$ by $f'(x)$or y'.

Example 2

Differentiate the following from first principles.

(a) $y = 9x + 5$

(b) $y = \frac{1}{x}$

(c) $y = \frac{1}{x^2}$

(d) $y = \sqrt{x}$

Solution

As x increases by δx, y will increase by δy

(a) $\Rightarrow y + \delta y = 9(x + \delta x) + 5$

$\delta y = 9x + 9\delta x + 5 - 9x - 5$

$= 9\delta x$

Dividing through by δx

$\Rightarrow \frac{\delta y}{\delta x} = 9$

As $\delta x \to o, \frac{\delta y}{\delta x} \to 0$

$\therefore \frac{dy}{dx} = 9$

(b) $\Rightarrow y + \delta y = \frac{1}{(x+\delta x)}$

$\Rightarrow \delta y = \frac{1}{(x+\delta x)} - \frac{1}{x}$

$= \frac{x-(x+\delta x)}{x(x+\delta x)}$

$= \frac{-\delta x}{x^2+x\delta x}$

Dividing through by δx, gives

$\frac{\delta y}{\delta x} = -\frac{1}{x^2+x\delta x}$

As $\delta x \to 0, \frac{\delta y}{\delta x} \to \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = -\frac{1}{x^2}$

(c) As x increases by δx, y will increase by δy

$\Rightarrow y + \delta y = \frac{1}{(x+\delta x)^2}$

$\delta y = \frac{1}{(x+\delta x)^2} - \frac{1}{x^2}$

$= \frac{x^2-(x^2+2x\delta x+(\delta x)^2)}{x^2(x^2+2x\delta x+(\delta x)^2)}$

$= \frac{x^2-x^2-2x\delta x-(\delta x)^2}{x^2(x^2+2x\delta x+(\delta x)^2)}$

$= \frac{-2x\delta x-(\delta x)^2}{x^2(x^2+2x\delta x+(\delta x)^2)}$

Dividing through by δx gives

$\frac{\delta y}{\delta x} = \frac{-2x-\delta x}{x^2(x^2+2x\delta x+(\delta x)^2)}$

As $\delta x \to 0, \frac{\delta y}{\delta x} \to \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{-2x}{x^4}$

$\therefore \frac{dy}{dx} = -\frac{2}{x^3}$

(d) Squaring both sides

$\Rightarrow y^2 = x$

$\Rightarrow (y + \delta y)^2 = x + \delta x$

$\Rightarrow y^2 + 2y\delta y + (\delta y)^2 = x + \delta x$

$\Rightarrow 2y\delta y + (\delta y)^2 = \delta x$

Since δy is very small then $(\delta y)^2 \approx 0$

$\Rightarrow 2y\delta y = \delta x$

Dividing through by δx

$\Rightarrow 2y\frac{\delta y}{\delta x} = 1$

$\Rightarrow \frac{\delta y}{\delta x} = \frac{1}{2\sqrt{x}}$

As $\delta x \to 0, \frac{\delta y}{\delta x} \to 0$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

Qn. Find from first principles the differential coefficient of y with respect to x if $f(x) = 4x + 2x^2$ and hence find $f'(2)$

Qn. Differentiate the following from first principles;

(a) $y = 3x^2$ (b) $y = 5x - x^2 + 7$ (c) $y = 2x^2 + 5x - 3$ (d) $y = x^2 + x^3$

(e) $y = \frac{1}{2x+3}$

Qn.If $y = \sqrt{x}$, show that $\frac{\delta y}{\delta x} = \frac{1}{\sqrt{(x+\delta x)}+\sqrt{x}}$ hence deduce $\frac{dy}{dx}$.

Note*: When differentiating a function we normally use the variable given as a respect in which the derivative is taken. If the function y is in terms of x we use $\frac{dy}{dx}$, if it is in terms of t we use $\frac{dy}{dt}$, if it is in terms of θ we use $\frac{dy}{d\theta}$ e.t.c*

Summary of results:

Finding the gradient of a curve simply implies differentiation and by differentiation the law states that **"Multiply the base by the power and subtract 1 from the power".**

Derivative of $y = ax^n$

- If $y = ax^n$, then $\frac{dy}{dx} = anx^{n-1}$ for all rational values of n.

Notice: If $y = a$, a constant, then this can be written as $y = ax^0$ and $\frac{dy}{dx} = (a \times 0)x^{-1} = 0$.

- In other words, the derivative of a constant is always zero. Thinking about this result geometrically, $y = a$ is a horizontal line which has a gradient of zero.

Example 3

Differentiate the following;

(i) $y = x^2$

(ii) $y = x^3$

Solution

(i) $\frac{dy}{dx} = 2x$

(ii) $\frac{dy}{dx} = 3x^2$

Example 4

Find $\frac{dy}{dx}$ for each of the following with respect too x

(a) $y = \frac{3}{4x^2}$ (b) $y = x^{\frac{1}{3}}$ (c) $y = \frac{1}{\sqrt{x}}$ (d) $y = \frac{1}{x^5}$

Solution

(a) $y = \frac{3}{4}x^{-2}; \frac{dy}{dx} = -\frac{6}{4}x^{-3} = -\frac{3}{2x^3}$

(b) $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

(c) $y = x^{\frac{-1}{2}}$; $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}$

(d) $y = x^{-5}$; $\frac{dy}{dx} = -5x^{-6} = -\frac{5}{x^6}$

Differentiation of a sum or difference of two functions

- When y comprises more than one function, to find the first derivative, we differentiate each function in turn. In other words, when $y = f(x) \pm g(x)$then $\frac{dy}{dx} = f'(x) \pm g'(x)$.
- A function may not be given in the form ax^n. In this case, it becomes necessary to manipulate the expression for y ad write it as a sum of functions, each in the form ax^n

Example 5

Find $f'(x)$ for each of the following;

(a) $f(x) = 4x + 1$ (b) $f(x) = 2x^3 + \sqrt{x}$(c)$g(x) = x + \frac{1}{x}$ (d) $f(x) = x^2 + 6x^{\frac{1}{3}} - 3$.

Solution

(a) $f'(x) = 4$

(b) $f(x) = 2x^3 + x^{\frac{1}{2}}$

$\Rightarrow f'(x) = 6x^2 + \frac{1}{2}x^{-\frac{1}{2}} = 6x^2 + \frac{1}{2\sqrt{x}}$

(c) $g(x) = x + \frac{1}{x} = x + x^{-1}$

$g'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2}$

(d) $f'(x) = 2x + \frac{6}{3}x^{-\frac{2}{3}} = 2x + \frac{2}{\sqrt[3]{x^2}}$

Example 6

Find $\frac{dy}{dx}$ for each of the following;

(a) $y = (x + 3)^2$ (b) $y = \frac{5x^3+3x^2}{x^2}$ (c) $y = x^2(4x - 2)$ (d) $y = (x - 3)(x + 1)$

(e) $y = \sqrt{x}(x^2 - 1)$

Solution

(a) $y = x^2 + 6x + 9$

$\therefore \frac{dy}{dx} = 2x + 6$

(b) $y = \frac{5x^3}{x^2} + \frac{3x^2}{x^2} = 5x + 3$

$\therefore \frac{dy}{dx} = 5$

(c) $y = 4x^3 - 2x^2$

$\therefore \frac{dy}{dx} = 12x - 4x$

(d) $y = x^2 - 2x - 3$

$\therefore \frac{dy}{dx} = 2x - 2$

(e) $y = x^{\frac{5}{2}} - x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

Second derivative:

- The derivative of $\frac{dy}{dx}$, that is $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is denoted as $\frac{d^2y}{dx^2}$ and is called the second derivative of y with respect to x.
- The derivative of $f'(x)$ is denoted by $f''(x)$ and is called the second derivative of $f(x)$ with respect to x.

Example 7

Given that $f(x) = x + \frac{1}{x}$ find $f'(x)$ and $f''(x)$

Solution

$\Rightarrow f(x) = x + \frac{1}{x} = x + x^{-1}$

$\Rightarrow f'(x) = 1 - x^{-2}$

$= 1 - \frac{1}{x^2}$

Given that $f'(x) = 1 - x^{-2}$

$\Rightarrow f''(x) = 2x^{-3}$

$= \frac{2}{x^3}$

Example 8

Find $\frac{d^2y}{dx^2}$ of the following

(a) $y = x^2 + 7x - 6$

(b) $y = x - 7x^4$

(c) $y = 5x - \frac{3}{\sqrt{x}}$

(d) $y = (x + 3)(x - 1)$

Solution

(a) $\frac{dy}{dx} = 2x + 7$

$\therefore \frac{d^2y}{dx^2} = 2$

(b) $\frac{dy}{dx} = 1 - 28x^3$

$\therefore \frac{d^2y}{dx^2} = -84x^2$

(c) $y = 5x - 3x^{-\frac{1}{2}}$

$\frac{dy}{dx} = 5 + \frac{3}{2}x^{-\frac{3}{2}}$

$\therefore \frac{d^2y}{dx^2} = -\frac{9}{4}x^{-\frac{5}{2}}$

(d) $y = x^2 + 2x - 3$

$\frac{dy}{dx} = 2x + 2$

$\therefore \frac{d^2y}{dx^2} = 2$

Example 9

If $f(x) = x^3 + 1$ find the value of a given that $f''(a) = 24$

Solution

$f'(x) = 3x^2$

$f''(x) = 6x$

Since $f''(a) = 24$

$\Rightarrow 6a = 24$

$\therefore a = 4$

Example 10

If $y = 4x^3$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence show that y satisfies the equation

$3y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 = 0$

Solution

$\Rightarrow \frac{dy}{dx} = 12x^2$ and $\frac{d^2y}{dx^2} = 24x$

$$\Rightarrow 3y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 = 3(4x^3)(24x) - 2(12x^2)^2$$
$$= 288x^4 - 288x^4$$
$$= 0$$

Exercise 8.0

1. Differentiate each of the following from first principles:

(a) $y = x^4$ (b) $y = 2x^2 + 3$ (c) $y = 1 - x^2$ (d) $y = x^3 - 6x$ (e) $y = \sqrt{2x}$

2. Find $\frac{dy}{dx}$ of the following with respect to x

(a) $y = \frac{1}{3}x^4$ (b) $y = \frac{3}{2x^2}$ (c) $y = \sqrt{x^5}$ (d) $y = -\frac{6}{\sqrt[3]{x}}$

3. Find $\frac{dy}{dx}$ of the following with respect to x

(a) $y = x^4 - 3x^2 + 2$ (b) $y = x + \frac{1}{x}$ (c) $y = \frac{3}{x} - 1 + 4x^3$ (d) $y = x^7 + 3x^4$

4. Differentiate the following with respect to x

(a) $2\sqrt{x} + 1$ (b) $\sqrt{x} + \frac{1}{\sqrt{x}}$ (c) $4x^{\frac{1}{2}} + 2x - 1$ (d) $\sqrt{x} + 1 + \frac{1}{\sqrt{x}}$

5. Find $f^1(x)$ for each of the following:

(a) $f(x) = 4x - 7$ (b) $f(x) = 2\sqrt{x} + \frac{5}{2x}$ (c) $f(x) = \frac{3}{\sqrt[6]{x}} - \frac{2}{\sqrt[4]{x}}$

6. Find $\frac{dy}{dx}$ for each of the following with respect to x

(a) $y = x^2(x + 2)$ (b) $y = \sqrt{x}(5 + x)$ (c) $y = (x + 3)(x - 2)$ (d) $y = (x - 2)^2$

7. Differentiate the following with respect to x

(a) $x^{\frac{-1}{2}}(x + 1)$ (b) $\frac{3x^3+5}{x^2}$ (c) $\frac{(x+5)^2}{x}$ (d) $\frac{x^2+5}{3\sqrt{x}}$ (e) $\frac{3\sqrt{x}-7}{2\sqrt{x}}$

8. Find $\frac{d^2y}{dx^2}$ for each of the following:

(a) $y = 3x^3 + 5x$ (b) $y = \frac{2}{x^2} - x$ (c) $y = (x^2 - 1)(2x + 3)$ (d) $y = x^2 - 4x^6$

(e) $y = \frac{1}{\sqrt{x}} - \sqrt{x}$ (f) $y = \sqrt[3]{x} - x^3$

9. Given that $y = \frac{1}{\sqrt{x}}$ show that $2x\left(\frac{d^2y}{dx^2}\right) + 3\frac{dy}{dx} = 0$
10. Given that $y = \frac{1}{x^2}$, show that $y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 - 10y^3 = 0$
11. Given that $y = ax^2 + bx$ and $\frac{d^2y}{dx^2} = 4\left(\frac{dy}{dx}\right)^2 - 32y$; find the possible values of the constants a and b.
12. Given that $y = x^4$, show that $\frac{4y}{3}\left(\frac{d^2y}{dx^2}\right) - \left(\frac{dy}{dx}\right)^2 = 0$
13. Given that $y = \frac{x+1}{x^2}$, show that $\frac{d^2y}{dx^2} + \frac{4}{x}\left(\frac{dy}{dx}\right) + \frac{2}{x^2}y = 0$

Tangent and Normal a curve

- The gradient of a curve at a point P is the gradient of the tangent to the curve at the point P.

Example 11

Find the gradient of the curve $f(x) = x^2 + \frac{1}{x}$ at the point P (1, 2)

Solution

$\Rightarrow f(x) = x^2 + \frac{1}{x}$

$= x^2 + x^{-1}$

$\Rightarrow f'(x) = 2x - x^{-1}$

$= 2x - \frac{1}{x^2}$

$\Rightarrow f'(1) = 2(1) - \frac{1}{(1)^2}$

$= 1$

$\therefore$the gradient of the curve at P(1, 2) is 1.

Example 12

The gradient of the curve $y = 3x^2 + x - 3$ at the point P is 13. Find the coordinates of point P.

Solution

When $y = 3x^2 + x - 3$, then $\frac{dy}{dx} = 6x + 1$

We are given that $\frac{dy}{dx} = 13$ at point P.

$\Rightarrow 6x + 1 = 13;\ \ x = 2$

$\Rightarrow y = 3(2)^2 + 2 - 3 = 11$

$\therefore$the coordinates of Pare (2, 11)

Example 13

The curve C is given by $y = ax^2 + b\sqrt{x}$, where a and b are constants. Given that the gradient of C at the point (1, 1) is 5, find a and b.

Solution

Rewriting $y = ax^2 + b\sqrt{x}$ as $y = ax^2 + bx^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = 2ax + \frac{b}{2}x^{-\frac{1}{2}} = 2ax + \frac{b}{2\sqrt{x}}$

Now the point (1, 1) lies on the curve C.

$\Rightarrow 1 = a(1)^2 + b(1)^{\frac{1}{2}}$

$\Rightarrow 1 = a + b; a + b = 1$……………….(1)

The gradient of the curve when $x = 1$ is 5.

$\Rightarrow 2(a) + \frac{b}{2\sqrt{1}} = 5$

$\Rightarrow 4a + b = 10$

(2)-(1) gives

$3a = 9; a = 3$

$\Rightarrow b = -2$

$\underline{\therefore a = 3 \; and \; b = -2}$

Equation of a tangent:

We now look at how to find the equation of the tangent line to the curve at a particular point.

Example 14

Find the equation of the tangent to the curve $y = x^2$ at the point P (3, 9)

Solution

When $y = x^2$; $\frac{dy}{dx} = 2x$

When $x = 3, \frac{dy}{dx} = 2(3) = 6$

$\Rightarrow y = 6x + c$

The tangent passes through the point P (3, 9)

$\Rightarrow 9 = 6(3) + c; c = -9$

$\underline{\therefore y = 6x - 9 \; is \; the \; equation \; of \; the \; tangent.}$

Example 15

Find the equation of the tangent to the curve $f(x) = \frac{1}{x^2}$ at the point P (-1, 1). Find the coordinates of the point where this tangent meets the curve again

Solution

$\Rightarrow f'(x) = -\frac{1}{x^3}$

At P $x = -1$

$\Rightarrow f'(-1) = -\frac{2}{(-1)^3} = 2$

$y = 2x + c$

$\Rightarrow 1 = 2(-1) + c; \;\; c = 3$

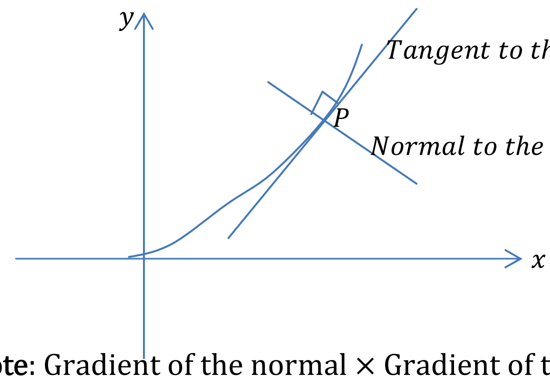
$\therefore y = 2x + 3$ is the tangent.

The tangent meets the curve again at the points whose x −coordinates satisfy

$2x + 3 = \frac{1}{x^2}$
$\Longrightarrow 2x^3 + 3x^2 - 1 = 0$
$\Longrightarrow (2x - 1)(x^2 + 2x + 1) = 0$
$\Longrightarrow x = \frac{1}{2}$ and $(x^2 + 2x + 1) = 0;\ \Leftrightarrow (x + 1)^2 = 0; x = -1$
When $x = \frac{1}{2}, f\left(\frac{1}{2}\right) = 4\ \ ;(1/2\,,4)$
When $x = -1; f(-1) = 1(1,1)$
$\therefore$the tangent meets the curve again at the point with coordinates $(1/2\,,4)$.

Equation of the normal

The normal to a curve at a point P is the straight line through P which is perpendicular to the tangent at P.



Note: Gradient of the normal $\times$ Gradient of the tangent $= -1$

Example 16

Find the equation of the normal to the curve $y = 3x^2 + 7x - 2$ at the point P where $x = -1$

Solution

When $x = -1, y = 3(-1)^2 + 7(-1) - 2 = -6; P(-1,-6)$
When $y = 3x^2 + 7x - 2; \frac{dy}{dx} = 6x + 7$
At P, $\frac{dy}{dx} = 6(-1) + 7 = 1$
Gradient of the tangent is 1 and the gradient of the normal at P is $-\frac{1}{1} = -1$
The equation of the normal at P is given by
$y + 6 = -1(x + 1)$
$\therefore y = -x - 7$

Example 17

Find the equation of the normal to the curve $y = x^3 - 8$ at the point where the curve cuts the y-axis.

Solution

The curve cuts the y-axis when $x = 0$, and $y = -8$; P$(0,-8)$ is the point

$\Rightarrow \frac{dy}{dx} = 3x^2$

At P ;$\frac{dy}{dx} = 3(0)^2 = 0$

Gradient of the normal is ∞

The equation of the normal is $y = -8$

Example 18

Find the equation of the normal to the curve $y = x^3 - 3x^2 + 4$ which are perpendicular to the line $y = 24x + 1$.

Solution

$y = 24x + 1$;has gradient 24

Gradient of the normal is $-\frac{1}{24}$

Gradient of the tangents is 24.

When $y = x^3 - 3x^2 + 4$; $\frac{dy}{dx} = 3x^2 - 6x$

$\Rightarrow 3x^2 - 6x = 24$

$\Rightarrow 3(x^2 - 2x - 8) = 0$

$\Rightarrow (x^2 - 4x + 2x - 8) = 0$

$\Rightarrow \left(x(x-4) + 2(x-4)\right) = 0$

$\Rightarrow (x+2)(x-4) = 0$

Either $x = -2, y = -16; P(-2, -16)$

Or $x = 4, y = 20; Q(2, 20)$

Both equations have the equation of the form $y = -\frac{1}{24}x + c$

At P(-2, -16); $-16 = -\frac{1}{24}(-2) + c; c = -\frac{193}{12}$ and $y = -\frac{1}{24}x - \frac{193}{12}$

At Q(4, 20); $20 = -\frac{1}{24}(4) + c; c = -\frac{121}{6}$ and $y = -\frac{1}{24}x + \frac{121}{6}$

$\therefore\ y = -\frac{1}{24}x - \frac{193}{12}andy = -\frac{1}{24}x + \frac{121}{6}$ are the two normal.

Exercise 8.1

1. Find the gradient of the curve $y = x^2 + 6\text{x} - 5$ at the point where the curve cuts the $y - axis$.
2. A curve has equation $y = Ax^3 + \text{B}x^2 + \text{Cx} + \text{D}$, where A, B, C and D are constants. Given that the curve has gradient -4 at the point (1, 2) and gradient 8 at the point (-1, 6), find A, B, C and D. $Ans(A = 2, B = -3, C = -4, D = 7)$
3. Find the equation of the tangent at the point (1,-1) to the curve $y = 2 - 4x^2 + x^3$.What are the coordinates of the point where the tangent meets the curve again? (ans: $y = -5x + 4, (2, -6), y = -4x + 2$)
4. Find the equation of the tangents to the curve $y = x^3 - 6x^2 + 12x + 2$Which are parallel to the line$y = 3x$
5. Find the coordinates of the points where the curve $y = x^2 - x - 12$ cuts th x-axis and determine the gradient of $y = x^2 - x - 12$ at these points.
6. The line $y = 4x - 5$ cuts the curve $y = x^2 - 2x$ at two points. Find the gradient of $y = x^2 - 2x$ at these two points.

7. Find the equation of the normal to the curve $y = (x^2 + x + 1)(x - 3)$at the point where it cuts the x-axis. $(x + 13y - 3) = 0$
8. Find the value of x for which the gradient function of the curve $y = 2x^3 + 3x^2 - 12x + 3$is zero. Hence find the equations of the tangents to the curve which are parallel to the x-axis
9. Find the equation of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at the point (1, 4). At what points of the curve is the tangent parallel to the line $4x + y - 3 = 0$
10. Find the equation of the normal to the curve $y = x^2$ at the point where $x = 2$.
11. Find the equation of the tangent and normal to the curve to the curve $y = x^2 - 4\text{x} + 1$ at the point (-2, 13)
12. The tangent to the curve $y = ax^2 + bx + 2$ at $\left(1, \frac{1}{2}\right)$ is parallel to the curve $y = x^2 + 6x + 10$at (-2, 2). Find the values of a and b.
13. (a) Find the equation of the two tangents to the curve $y = x^2 - 5x + 4$ at the points where the curve cuts the $x - axis$.

(b) Find also the coordinates of the point of intersection of the two tangents.

$Ans((a)y = -3x + 3, y = 3x - 12\ (b)(5/2\,, 9/2)$

14. T is the tangent to the curve $y = x^2 + 6x - 4$at (1, 3) and N is the normal to the curve $y = x^2 - 6x + 18$ at (4, 10). Find the coordinates of the point of intersection of T and N.
15. The gradient of $y = x^2 - 4x + 6$ at (3, 3) is the same as the gradient of $y = 8x - 3x^2$at (a, b). Find the values of a and b.
16. The curve $y = ax^3 - 2x^2 - x - 7$has gradient of 3 at the point where x=2. Determine the value of a.
17. The normal to the curve $y = x^2 - 4x$ at the point (3, -3) cuts the x-axis at A and the y-axis at B. find the equation of the normal and the coordinates of A and B.
18. The tangent to the cure $y = 2x^2 + ax + b$ at the point (-2, 11) is perpendicular to the line$2y = x + 7$. Find the value ofa and b.
19. The curve $y = ax^2 + bx + c$ has a maximum point at (2, 18) and passes through the point (0, 10). Evaluate a, b and c.
20. The gradient of the curve $y = 3x^2 + x - 3$ at the point P is 13. Find the coordinates of P. $\boldsymbol{Ans}(\mathbf{2}, \mathbf{11})$
21. Find the equation of the normal to the curve $y = 3x^2 + 7x - 2$ at the point where $x = -1$. $\boldsymbol{Ans}(\boldsymbol{y} = -\boldsymbol{x} - \mathbf{7})$
22. (a) Find the equation of the normal at the point (2, 3) on the curve $y = 2x^3 - 12x^2 + 23x - 11$.

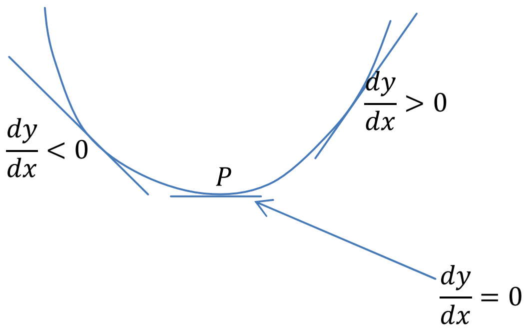
    (b) Find also the coordinates of the point where the normal meets the curve again.
23. (a) Find the equation of the tangent at the point (1, 2) on the curve $y = x^3 + 3x - 2$.

    (b) Find also the coordinates of the point where this tangent meets the curve again.
24. Find the coordinates of the two points where the normal to the curve $y = \frac{1}{2}x^3 - x + 3$at the point (0, 3) meets the curve again. $\boldsymbol{Ans}((-\mathbf{2}, \mathbf{1}), (\mathbf{2}, \mathbf{5}))$
25. The normal to the curve $y = x^2 + 5x - 2$ at the point where $x = -3$, and the tangent to the same curve at the point where $x = 1$, meet at the point Q. Find the coordinates of Q. $Ans(-1/3\,, -16/3)$

26. The normal to the curve $y = x^3 + cx$ at the point (2, d) has gradient1/2. Find the values of the constants c and d. $Ans(c = -14, d = -20)$

Maximum, Minimum and point of inflexion

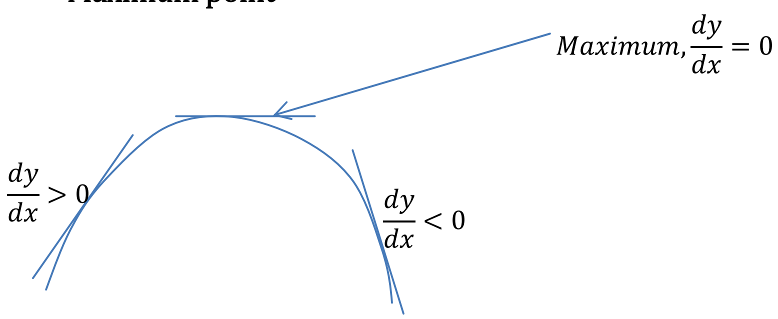
- A point on a curve at which the gradient is zero, i.e. where $\frac{dy}{dx} = 0$ is called **astationary/ turning point.**
- At a stationary point, the tangent to the curve is horizontal and the curve is "flat".
- There are three types of stationary point and we must know how to distinguish one from another.
- **Minimum point**



In this case, the gradient of the curve is negative to the left of point P. To the right of point P, the gradient of the curve is negative.

To the left of P	At point P	To the right of P
$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$
-	0	+

- **Maximum point**

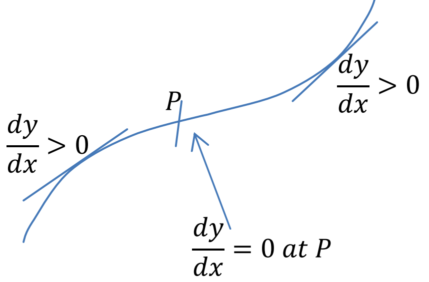


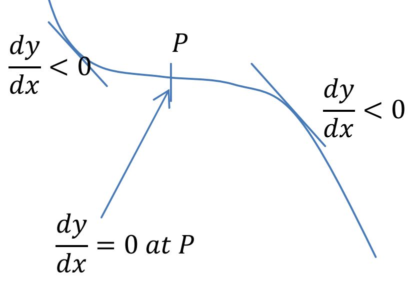
- In this case, the gradient of the curve is positive to the left of point P. To the right of point P, the gradient of the curve is negative.

To the left of P	At point P	To the right of P
$\frac{dy}{dx} > 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0$
+	0	-

- **Point of inflexion**

Case I **Case II**





- In this case, the gradient has the same sign each side of the stationary point.
  Case I

To the left of P	At point P	To the right of P
$\frac{dy}{dx} > 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$
+	0	+

Case II

To the left of P	A point P	To the right of P
$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} < 0$
-	0	-

- The above is called the table method of distinguishing turning points.

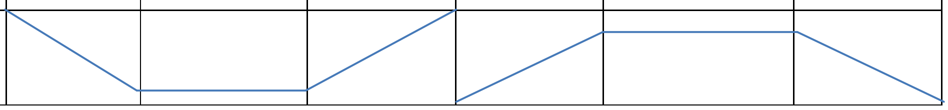
Note: The maximum and minimum points can be called the greatest or least values where necessary.

Example19

Find the coordinates of the stationary points on the curve $y = x^3 + 3x^2 + 1$ and distinguish their nature.

Solution

When $y = x^3 + 3x^2 + 1$, then $\frac{dy}{dx} = 3x^2 + 6x$
At stationary point, $\frac{dy}{dx} = 0$
$\Longrightarrow 3x^2 + 6x = 0$
$\Longrightarrow 3x(x + 2) = 0; x = 0, x = -2$
When $x = 0, y = (0)^3 + 3(0)^2 + 1 = 1$
When $x = -2, y = (-2)^3 + 3(-2)^2 + 1 = 5$
The coordinates of the stationary points are (0, 1) and (-2, 5)
Distinguishing between them;

Value of x	L	$x = 0$	R	L	$x = -2$	R
Sign of $\frac{dy}{dx} = 3x^2 + 6$	-	0	+	+	0	-
						

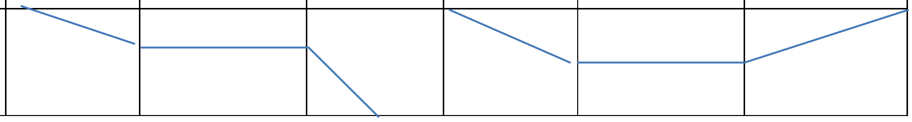
The stationary point (0, 1) is a maximum
The stationary point (-2, 5) is a maximum.

Example 20

Find the coordinates of the stationary points on the curve $y = x^4 - 4x^3$
Solution
$\frac{dy}{dx} = 4x^3 - 12x^2$
$\Longrightarrow 4x^3 - 12x^2 = 0$
$\Longrightarrow 4x^2(x - 3) = 0$
$\Longrightarrow x = 0, x = 3$
When $x = 0, y = 0; (0, 0)$
When $x = 3, y = (3)^4 - 4(3)^3 = -27; (3, -27)$

Value of x	L	$x = 0$	R	L	$x = 3$	R
Sign of	-	0	-	-	0	+

$\frac{dy}{dx} = 4x^3 - 12x^2$						



The point (0, 0) is a point of inflexion

The point (3, -27) is a minimum.

Using the second derivative test

For a maximum $\frac{d^2y}{dx^2} < 0$ (negative)

For a minimum $\frac{d^2y}{dx^2} > 0$ (positive)

For the point of inflexion $\frac{d^2y}{dx^2} = 0$ (no sign change at all)

Example 21

Find the coordinates of the stationary points on the curve
$f(x) = x^3 - 6x^2 - 15x + 1$, using the second derivative determine their nature.
Solution
When $f(x) = x^3 - 6x^2 - 15x + 1$
$\Rightarrow f^1(x) = 3x^2 - 12x - 15$
$\Rightarrow 3x^2 - 12x - 15 = 0$
$\Rightarrow 3(x-5)(x+1) = 0$
Solving gives $x = 5$ *or* $x = -1$
When $x = 5, f(5) = (5)^3 - 6(5)^2 - 15(5) + 1 = -99; (5, -99)$
When $x = -1, f(-1) = (-1)^3 - 6(-1)^2 - 15(-1) + 1 = 9; (-1, 9)$
The stationary points are (5, -99) and (-1, 9)
The second derivative is given by $f''(x) = 6x - 12$
At the point (5, -99), $f''(5) = 6(5) - 12 = 18 > 0$
∴The stationary point (5, -99) is a minimum.
At the point (-1, 9), $f''(-1) = 6(-1) - 12 = -18 < 0$
∴The stationary point (-1, 9) is a maximum.
Qn. Using the second derivative test find the greatest or least value of y on the following cases;

(a) $y = 4x - 2x^2$
(b) $y = 3x^2 + 2x - 5$
(c) $4x^2 - 6x + 2$

Exercise 8.2

1. Find the coordinates of the points on each of the following curves at which the gradient is zero.
(a) $y = x^2 - 4x + 3$
(b) $y = 6 + 9x - 3x^2 - x^3$
(c) $y = x^3 - 6x^2 - 36x$
(d) $y = 1 - 6x + 6x^2 + 2x^3 - 3x^4$
2. Find the coordinates of the stationary points on each of the following curves, and determine their nature.
(a) $y = x^2 - 2x + 5$
(b) $y = (x - 4)(x - 2)$
(c) $y = x^4 - 2x^2 + 3$
(d) $y = x^3 - 5x^2 + 3x + 1$
(e) $y = x^4 - 14x^2 + 24x - 10$
3. Find the coordinates of the stationary points on each of the following curves, and determine their nature.
(a) $y = x + \frac{1}{x}$
(b) $y = x^2 + \frac{16}{x}$
(c) $y = \frac{12x^2 - 1}{x^3}$
(d) $y = \frac{2}{x^3} - \frac{1}{x^2}$
4. Find the greatest or least value of y in the following cases
(a) $y = 4x - 2x^2$
(b) $y = 3x^2 + 2x - 5$
(c) $y = 4x^2 - 6x + 2$
(d) $y = 4x - x^2$
(e) $y = x^2 + 4x + 3$
5. Find the turning values of y on the graph $y = f(x)$ where $f(x) = 5 + 24x - 9x^2 - 2x^3$ and distinguish between them.
6. The equation of the curve is given by $y = 3x^4 - 12x^2 + 5$. Determine the minimum and maximum values of y.

8.3 Curve Sketching

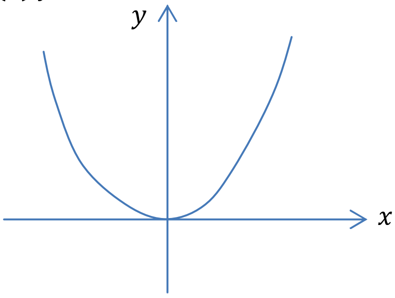
Procedure

The following steps are required in order to come up with a smart curve.

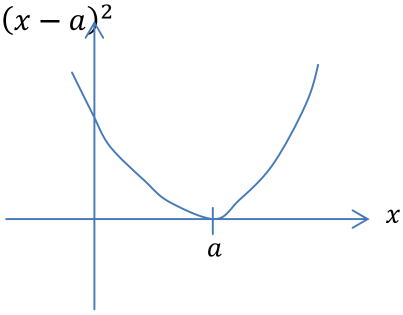
(i) Obtain where the curve crosses the y-axis i.e. when $x = 0$
(ii) Obtain where the curve crosses the y-axis i.e. when $y = 0$
(iii) Obtain the stationary/turning points of the curve i.e. when $\frac{dy}{dx} = 0$
(iv) Investigate the nature of the turning point.
(v) Hence sketch the curve.

The following are the obvious curves;

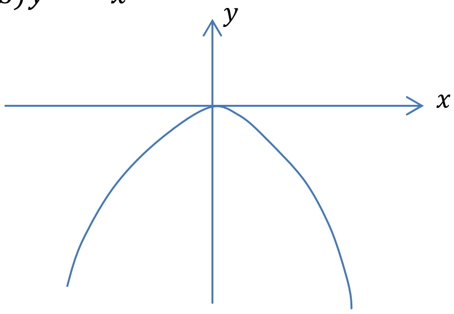
(a) $y = x^2$



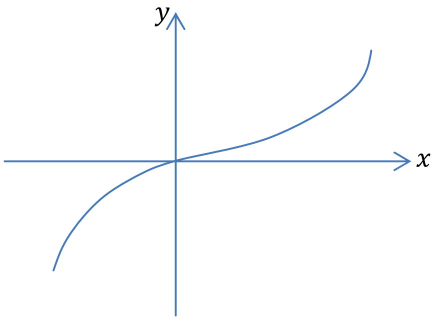
(d) $y = (x - a)^2$



(b) $y = -x^2$



(c) $y = x^3$



Example 22

Sketch the following curves;

(a) $y = 4x - x^2$
(b) $y = 4x^3 - 3x^4$

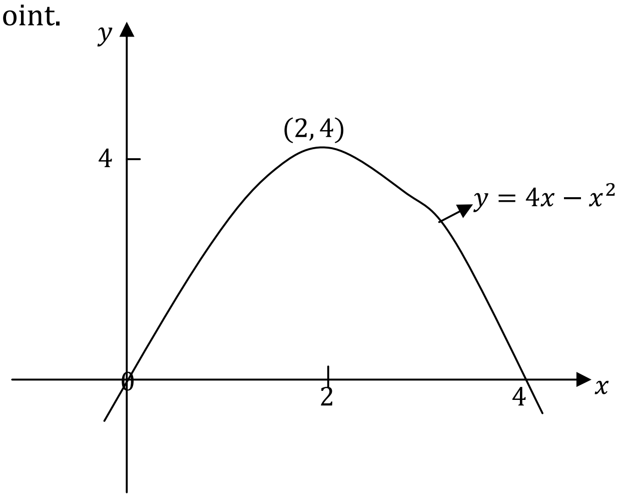
Solution

(a) <u>intercepts:</u>
when $x = 0, y = 0; (0, 0)$
when $y = 0, 0 = 4x - x^2$
$\Rightarrow x(4 - x) = 0;\ x = 0; (0, 0)$
$x = 4; (4, 0)$
<u>Turning points</u>
$\frac{dy}{dx} = 4 - 2x$

For turning point, $\frac{dy}{dx} = 0; 4 - 2x = 0, x = 2 \implies y = 4(2) - (2)^2 = 4; (2, 4)$

Nature of turning point using 2nd derivative test

$\frac{d^2y}{dx^2} = -2 > 0 \therefore (2, 4)$ is maximum point.



(b) $y = 4x^3 - 3x^4$

<u>Intercepts</u>

When $x = 0, y = 0;\ (0, 0)$ When $y = 0, 4x^3 - 3x^4 = 0; x^3(4 - 3x) = 0$

$\implies x = 0, x = 4/3\ \ (0, 0), (4/3, 0)$

<u>Turning points</u>

$\frac{dy}{dx} = 12x^2 - 12x^3$

$\implies 0 = 12x^2 - 12x^3; 0 = 12x^2(1 - x)$

$\implies x = 0, x = 1;$

When $x = 0, y = 0; (0, 0)$

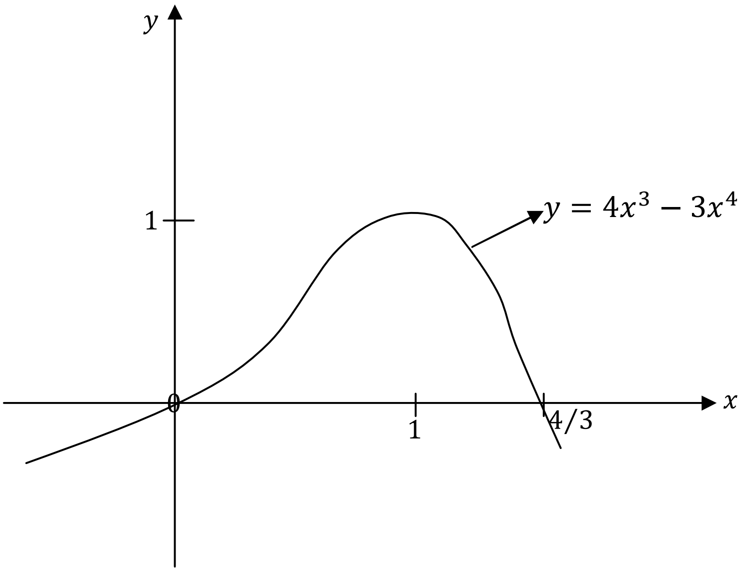
When $x = 1, y = 4(1)^3 - 3(1)^4 = 1; (1,1)$

Nature of turning points using 2nd derivative test

$\frac{d^2y}{dx^2} = 24x - 36x^2$

When $x = 0, \frac{d^2y}{dx^2} = 0, (0, 0)$ point of inflexion

When $x = 1, \frac{d^2y}{dx^2} = 24(1) - 36(1)^2 = -12 < 0; (1, 1)$ is a maximum



Note: We can use the <u>intercepts</u> and <u>behaviour of the curve at infinity</u> and sketch some curves without necessarily obtaining the turning points.

Example 23

Sketch the curve $y = (x + 1)(x - 1)(2 - x)$

Solution

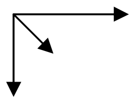
Intercepts:

When $x = 0, y = -2; (0, -2)$

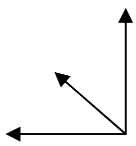
When $y = 0, x = -1, x = 1, x = 2; (-1, 0), (1, 0), (2, 0)$

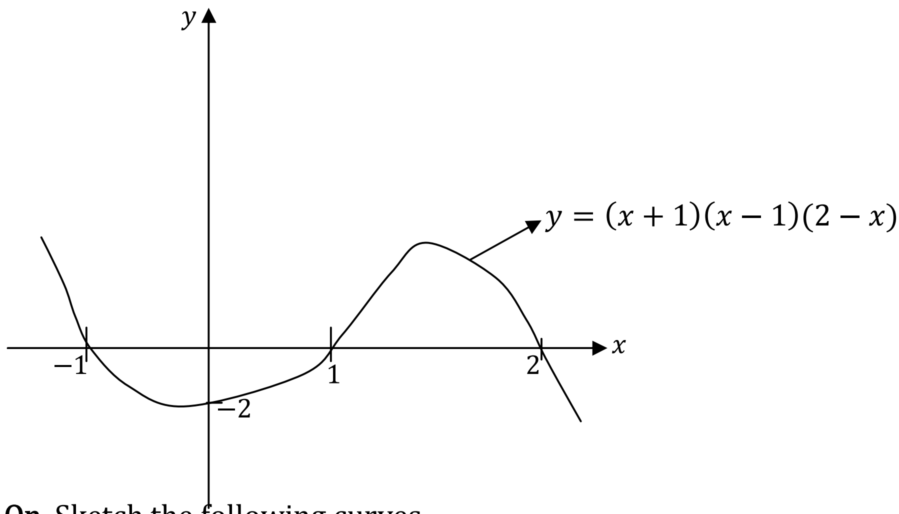
Behaviour at infinity.

AS x tends to $+\infty, y\ tends\ to - \infty$



AS x tends to $-\infty, y\ tends\ to + \infty$





Qn. Sketch the following curves

(a) $y = x^2(x - 2)^2$
(b) $y = (x + 1)^2(2 - x)$

Exercise 8.3

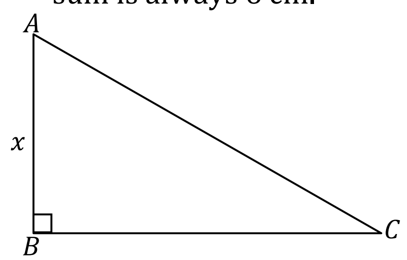
1. Sketch the following curves;
   (a) $y = x^2 - x - 2$ (b) $y = 15 + x - x^2$ (c) $y = (2x + 3)(x - 2)$ (d) $y = x(4 - x)$ (e) $y = 4x - 3x^3$ (f) $y = 5x^3 - 27x^2$
2. Determine the turning points of the curve $y = \frac{x^3}{3} - x$ and their nature. Hence sketch the curve.
3. Given the curve $y = x^3 - 4x + 2$, determine the turning points of the curve and their nature. Hence sketch the curve.
4. Given the curve $y = 3x^3 - 4x^2 - x$
   (a) Find the turning points of the curve
   (b) Distinguish between the nature of the turning points.
   (c) Hence sketch the curve.
5. Determine the nature of the turning points on the curve $y = 2x(x - 1)^2$. Hence sketch the curve.
6. Sketch the curve $y = 4 - x^2$
   (a) Find the coordinates of the point of intersection of the tangent to the curve at (2, 0) and the y-axis.
7. The function $y = ax^2 + bx + c$ has turning points at (0, 4) and (-1, 5).
   (i) find the values of a, b and c
   (ii) sketch the curve.

Applications of maximum and minimum to practical problems:

These are situations where we need to apply the idea of maximum and minimum to solve daily challenges.

Example 24:

In a right angled triangle ABC shown below, the length AB and BC vary such that their sum is always 6 cm.



(a) If the length of AB is x cm, write down, in terms of x, the length BC.
(b) Find the maximum area of the triangle ABC.

Solution

(a) $\overline{AB} + \overline{BC} = 6$

$\Longrightarrow x + \overline{BC} = 6;\ \ \overline{BC} = 6 - x$

(b) $A = \frac{1}{2}(\overline{AB})(\overline{BC}) = \frac{1}{2}(x)(6 - x) = 3x - \frac{1}{2}x^2$

$\frac{dA}{dx} = 3 - x;\ \ 0 = 3 - x;\quad x = 3$

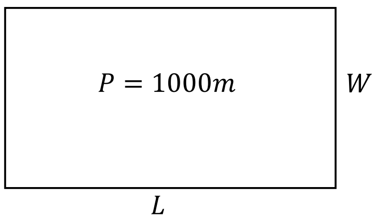
Using the second derivative test $\frac{d^2A}{dx^2} = -1 < 0; x = 3\ gives\ a$ max $area$

$\therefore A_{max} = \frac{1}{2}(3)(6 - 3) = 4.5cm^2$

Example 25:

1000m of fencing wire is to be used to make a rectangular enclosure. Find the greatest possible area, and the corresponding dimensions.

Solution



$\Longrightarrow P = 2(L + W)$

$\Longrightarrow 1000 = 2(L + W); L = 500 - W$

$A = LW = (500 - W)W = 500W - W^2$

$\Longrightarrow \frac{dA}{dW} = 500 - 2W$

$\Longrightarrow 0 = 500 - 2W; W = 250m, L = 250m$

Using the second derivative test $\frac{d^2A}{dx^2} = -2 < 0; W = 250\ gives\ a$ greatest $area$

$\therefore A_{greatest} = (250)(250) = 62500m^2$ and the dimensions are $250m \times 250m$.

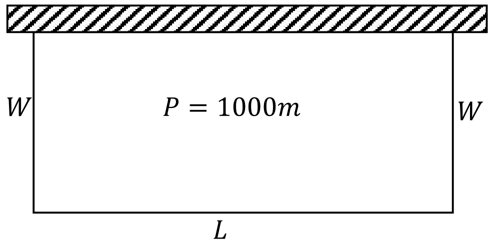
So it is a square that makes the greatest area from the perimeter of 1000m.

Example 26:

A rectangular sheep pen is to be made out of 1000m of fencing wire using an existing straight edge, find the maximum area possible and the dimensions necessary to achieve this.

Solution

Existing straight edge



$\Longrightarrow P = L + W + W$

$\Longrightarrow 1000 = L + W + W$

$\Longrightarrow L = 1000 - 2W$

$\Longrightarrow A = LW = (1000 - 2W)W = 1000W - 2W^2$

$\Longrightarrow \frac{dA}{dW} = 1000 - 4W$

$\Longrightarrow 0 = 1000 - 4W; W = 250m, L = 1000 - 2(250) = 500m$

Using the second derivative test $\frac{d^2A}{dx^2} = -4 < 0; W = 250\ gives\ a$ greatest $area$

$\therefore A_{greatest} = (500)(250) = 125000m^2$ and the dimensions are $500m \times 250m$.

Example 27:

A closed, right circular cylinder of base radius r cm and height h cm has a volume of $54\pi\ cm^3$. Show that S, the total surface area of the cylinder is given by $S = \frac{108\pi}{r} + 2\pi r^2$. Hence find the radius and height which make the surface area a minimum.

Solution

The volume, V is given by $V = \pi r^2 h$

$\Longrightarrow 45\pi = \pi r^2 h;\ \ h = \frac{54}{r^2}$

The total surface area, S is given by $S = 2\pi rh + 2\pi r^2$

$\Longrightarrow\ S = 2\pi r\left(\frac{54}{r^2}\right) + 2\pi r^2$

$\therefore\ S = \frac{108\pi}{r} + 2\pi r^2$ ....■

Maximum/minimum surface area occurs when $\frac{dS}{dr} = 0$

$\Longrightarrow S = 108\pi r^{-1} + 2\pi r^2$

$\Longrightarrow \frac{dS}{dr} = -108\pi r^{-2} + 4\pi r = -\frac{108\pi}{r^2} + 4\pi r$

$\Longrightarrow 0 = -\frac{108\pi}{r^2} + 4\pi r$

$\Longrightarrow 4\pi r^3 = 108\pi\ ; r^3 = 27, r = 3\ cm$

$\Longrightarrow h = \frac{54}{3^2} = 6cm$

Using the 2^{nd} derivative test

$\Longrightarrow \frac{d^2S}{dr^2} = 216\pi r^{-3} + 4\pi$

When $r = 3$

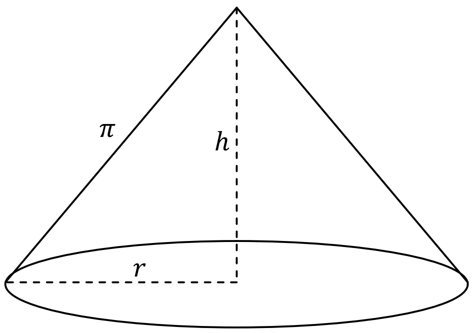
$\Rightarrow \frac{d^2S}{dr^2} = \frac{216\pi}{3^3} + 4\pi = 12\pi > 0$

$\therefore r = 3cm \ and \ h = 6cm$ give the minimum surface area of the cylinder.

Example 28

A right circular cone is to be made such that its slant height is π metres. Show that the maximum volume of the cone is $\frac{2\pi^4}{9\sqrt{3}}$

Solution



$\Rightarrow \pi^2 = r^2 + h^2;\ r^2 = \pi^2 - h^2$

But Volume, $V = \frac{1}{3}\pi r^2 h$ so $V = \frac{\pi}{3}h(\pi^2 - h^2) = \frac{\pi}{3}(\pi^2 h - h^3)$

$\Rightarrow \frac{dV}{dh} = \frac{\pi}{3}(\pi^2 - 3h^2)$

But for max volume, $\frac{dV}{dh} = 0$

$\Rightarrow \frac{\pi}{3}(\pi^2 - 3h^2) = 0;\ h = \frac{\pi}{\sqrt{3}}$

$\therefore$ Maximum Volume, $V = \frac{\pi}{3}\left(\pi^2 - \frac{\pi^2}{3}\right)\frac{\pi}{\sqrt{3}} = \boldsymbol{\frac{2\pi^4}{9\sqrt{3}}}$......................■

Exercise 8.3a

1. A farmer has 100m of metal railing with which to form two adjacent sides of a rectangular enclosure, the other two sides are being existing wall of the yard, meeting at right angles. What dimensions of the wall will give him the maximum possible area.
2. An open metal tank with a square base is made from $12m^3$ of sheet metal. Find the length of the side of the base for the volume of the tank to be maximum and find this value. $\boldsymbol{Ans}(\boldsymbol{2m}, \boldsymbol{4m^3})$
3. If $A = xy$ and $2x + 5y = 100$, find the maximum value of A and the values of x and y which gives this maximum value.
4. If $V = 4rx + 2r^2$ and $3r + x = 5$, find the maximum value of V and the values of r and x that gives this maximum value.
5. An aero plane flying level at 250m above the ground suddenly swoops down to drop supplies, and then regains its former altitude. It is h m above the ground t s after beginning its dive, where $h = 8t^2 - 80t + 250$. Find its least altitude during this operation and the interval of time during which it was losing height.

6. A rectangular block has a base $x\ cm$ square. Its total surface area is $150cm^3$. Prove that the volume of the block is $\frac{1}{2}(75x - x^3)cm^3$. Calculate;
(a) The dimensions of the block when its volume is maximum.
(b) This maximum volume
(c) Show that your answer is a maximum rather than a minimum.
7. A cylindrical can is made so that the sum of its height and the circumference of its base is $45\pi cm$. Find the radius of the base of the cylinder if the volume is a maximum.
8. A company manufacturing dog food wishes to pack the food in closed cylindrical tins. What should be the dimensions of each tin if each is to have a volume of $128\pi\ cm^3$ and the minimum possible area? $\boldsymbol{Ans(r = 4cm, h = 8cm)}$
9. A cylinder of volume V is to be cut from a solid sphere of radius R. Prove that the maximum value of V is $\frac{4\pi R^3}{3\sqrt{3}}$
10. A solid right circular cylinder of height h and radius r has a total external surface area of $600cm^2$.
    (a) Show that $h = \frac{300}{\pi r} - r$ and hence express the volume, V, in terms of r.
    (b) If h and r can vary, find $\frac{dV}{dr}$ and $\frac{d^2V}{dr^2}$ in terms of r. Show that V has a maximum value and find the corresponding value of r in terms of π
    (c) Calculate the ratio $h\!:\!r$ in this maximum case.
11. A box without a top is to be made from a thin sheet of metal of volume $288cm^3$. The base of the box will be rectangular with its length twice the breadth. Determine the dimensions of the box for which its surface area will be maximum. Hence find the surface area.
12. An industry is to produce closed cylindrical cans of radius r cm and height h cm. If each can is to have a total surface area of $96\pi cm^2$, find the;
    (i) expression for h in terms of r.
    (ii) expression for the volume V cm^3 in terms of r.
    (iii) value of r that gives the maximum volume of each can. Hence find the maximum volume
13. The profit, £y, generated from the sales of x items of a certain luxury product is given by the formula $y = 600x + 15x^2 - x^3$. Calculate the value of x which gives a maximum profit, and determine that maximum profit. $\boldsymbol{Ans(20, £10,000)}$
14. At a speed of x mph a certain car will travel y miles on each gallon of petrol, where $y = 15 + x - \frac{x^2}{110}$. Calculate the speed at which the car should aim to travel in order to maximize the distance it can cover on a single tank of petrol. $\boldsymbol{Ans(55mph)}$
15. A ball is thrown vertically upwards. AT time t seconds after the instant of projection, its height , y metres above the point of projection, is given by the formula $y = 15t - 5t^2$. Calculate the time at which the ball is at its maximum height, and find the value of y at that time. $\boldsymbol{Ans(1.5s, 11.25m)}$

16. A closed cuboidal box of square base has volume $8m^3$. Given that the square base has sides of length x metres, find the expression, in terms of x, for;

(a) The height of the box

(b) The surface area of the box

Given that the area enclosed is a minimum

(c) Find the value of x.

8.4 Velocity and Acceleration

The quantities displacement (r) or (s), the velocity (v) and acceleration (a) with respect to time (t) can be related using differentiation.

$\Longrightarrow \frac{ds}{dt} = v, \frac{dv}{dt} = a$

$\Longrightarrow \frac{d\boldsymbol{s}}{dt} = \boldsymbol{v}, \frac{d\boldsymbol{v}}{dt} = \boldsymbol{a}$ where **s**, **v** and **a** are given vectors

- Average velocity $= \frac{total\ distance\ travellwd}{total\ time\ taken} = \frac{increase\ in\ S}{increase\ in\ t}$
- The distance moved during the nth second is given by $S_n - S_{n-1}$
- The velocity moved during the nth second is given by $V_n - V_{n-1}$
- The acceleration moved during the nth second is given by $a_n - a_{n-1}$
- Average acceleration $= \frac{increase\ in\ v}{increase\ in\ t}$.

Example 29:

A stone is thrown vertically downwards from the top of a cliff, and the depth below the top, S m, after t s, is given by the formula $S = 2t + 4.9t^2$.

(a) Where is the stone after 1, 2, 3, 4?

(b) What is its velocity at these times?

(c) What is its average velocity during the 3rd second?

Solution

(a) Using $S = 2t + 4.9t^2$

$\Longrightarrow S = 2(1) + 4.9(1)^2 = 6.9m$

$\Longrightarrow S = 2(2) + 4.9(2)^2 = 23.6m$

$\Longrightarrow S = 2(3) + 4.9(3)^2 = 50.1m$

$\Longrightarrow S = 2(4) + 4.9(4)^2 = 86.4m$

(b) $V = \frac{dS}{dt} = 2 + 9.8t$

$\Rightarrow V = 2 + 9.8(1) = 11.8ms^{-1}$

$\Rightarrow V = 2 + 9.8(2) = 21.6ms^{-1}$

$\Rightarrow V = 2 + 9.8(3) = 31.4ms^{-1}$

$\Rightarrow V = 2 + 9.8(4) = 41.2ms^{-1}$

(c) Average velocity $= \frac{Total\ displacement}{Total\ time} = \frac{S_3 - S_2}{3-2} = \frac{50.1-23.6}{1} = 26.5ms^{-1}$

Example 30:

A particle moves such that its position vector at time t is $\boldsymbol{r}$. Find vector expressions for its velocity and acceleration at time t given that $\boldsymbol{r} = 3t\boldsymbol{i} - 2\boldsymbol{j} + 2t^3\boldsymbol{k}$.

Solution

$\Rightarrow \boldsymbol{v} = \frac{dr}{dt} = 3\boldsymbol{i} + 6t^2\boldsymbol{k}$

$\Rightarrow \boldsymbol{a} = \frac{dv}{dt} = 12t\boldsymbol{k}$

Exercise 8.4

1. A particle moves along a straight line in such a way that its distance from a fixed point O on the line after t s is s m, where $s = \frac{1}{6}t^4$. Find;
(a) Its velocity after 3 s and after 4 s.
(b) Its average velocity during the 4th second.
(c) Its acceleration after 2 s and after 4 s.
(d) Its average acceleration from t=2 to t=4.
$\boldsymbol{Ans}((\boldsymbol{a})\mathbf{18}, \mathbf{42}\tfrac{2}{3}, (\boldsymbol{b})\mathbf{13.5}, \mathbf{42}\tfrac{2}{3}, \mathbf{29}\tfrac{1}{6}(\boldsymbol{c})\mathbf{8}, \mathbf{32}\ (\boldsymbol{d})\mathbf{5}\tfrac{1}{3}, \mathbf{42}\tfrac{2}{3}, \mathbf{18}\tfrac{2}{3})$
2. A car starts from rest and moves a distance s m in t s, where $= \frac{1}{6}t^3 + \frac{1}{4}t^2$. What is the initial acceleration and the acceleration at the end of 2nd second? $\boldsymbol{Ans}\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{2.5}\right)$
3. A particle moves such that its position vector at time t is $\boldsymbol{r}$. Find vector expressions for its velocity and acceleration at time t given that $\boldsymbol{r} = (t+1)\boldsymbol{i} + t^4\boldsymbol{j} - 6t\boldsymbol{k}$.
4. If $v = 2t^3 - 9t^2$, find the value of t for which $a = 0$
5. A car starts from rest and moves a distance S m in t s, where $S = \frac{1}{6}t^3 + \frac{1}{4}t^2$. What is the initial acceleration and the acceleration at the end of the 2nd second? $\boldsymbol{Ans}\left(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{2.5}\right)$
6. A stone is thrown vertically upwards at 35ms-1. It is S m above the point of projection t s later, where $S = 35t - 4.9t^2$.
   (a) What is the distance moved and the average velocity during the 3rd second?
   (b) Find the average velocities for the intervals $t = 2$ to $t = 2.5$, $t = 2$ to $t = 2.1$.

7. A ball is thrown vertically upwards and its height after t s is S m where $S = 25.2t - 4.9t^2$. Find;
   (a) Its height and velocity after 3 s.
   (b) When it is momentarily at rest,
   (c) The greatest height reached
   (d) The distance moved in the 3rd second.
   (e) The acceleration when $t = 2\frac{4}{7}$

8.5 Composite Functions

The chain rule:

The process of differentiating a function which is a function of another variable is known as **chain rule.**

In this case we first let the inside function to be another variable before we can be able to differentiate.

Suppose y is a function of t, and t is itself a function of x, if $\delta y, \delta t \ and \ \delta x$ are corresponding small increments in the variable y, t and x then,

$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta t} \times \frac{\delta t}{\delta x}$ ....*(i)*

When $\delta y, \delta t \ and \ \delta x$ tend to zero, then

$\frac{\delta y}{\delta x} \to \frac{dy}{dx}, \ \frac{\delta y}{\delta t} \to \frac{dy}{dt}, \ \frac{\delta t}{\delta x} \to \frac{dt}{dx}$

And equation (i) becomes $\boldsymbol{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$ then this the **chain rule.**

Another important fact is the $\boldsymbol{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}$

Example 31

If $y = (3x - 1)^7$ find $\frac{dy}{dx}$

Solution

Let $u = 3x - 1$, then $y = u^7$

$\Rightarrow \frac{du}{dx} = 3$ and $\frac{dy}{du} = 7u^6$

Using chain rule;

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= (7u^6)(3) = 21u^6$

$\therefore \frac{dy}{dx} = 21(3x-1)^6$

Generally, using the function notation $y = f(g(x)), then\ \frac{dy}{dx} = f'(g(x)).g'(x)$

In example 30, this can be written as $y = (3x-1)^7$

$\Rightarrow \frac{dy}{dx} = 7(3x-1)^6(3) = 21(3x-1)^6$

We can remember this from the slogan " ***drop the power, subtract1 from the power and multiply it with the derivative of the bracket***"

Example 32

Find $\frac{dy}{dx}$ for each of the following functions

(a) $y = 2(1-x)^5$ (b) $y = (x^2+3)^4$ (c) $y = \frac{1}{(3-7x)}$ (d) $y = \sqrt{6x+1}$

Solution

(a) $\frac{dy}{dx} = 2(5)(1-x)^4(-1) = -10(1-x)^4$

(b) $\frac{dy}{dx} = 4(x^2+3)^3(2x)$

(c) $y = \frac{1}{(3-7x)} = (3-7x)^{-1}$

$\Rightarrow \frac{dy}{dx} = (-1)(3-7x)^{-2}(-7) = \frac{7}{(3-7x)^2}$

(d) $y = \sqrt{6x+1} = (6x+1)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{2}\right)(6x+1)^{\frac{-1}{2}}(6) = \frac{3}{\sqrt{(6x+1)}}$

Exercise 8.5

1. Find $\frac{dy}{dx}$ for each of the following functions.

(a) $y = (3x+2)^4$ (b) $y = (x^2+3x)^7$ (c) $y = \frac{1}{1+\sqrt{x}}$ (d) $y = \sqrt{1+x^2}$ (e) $y = \sqrt[3]{1-\sqrt{x}}$ (f) $y = (2-x^2)^5$ (g) $y = \frac{1}{\sqrt{5-2x}}$

2. Differentiate the following with respect to x.

(a) $f(x) = \sqrt{4x-5}$ (b) $f(x) = \left(5-4\sqrt{x}\right)^5$ (c) $f(x) = \frac{1}{4-\sqrt[3]{x}}$ (d) $f(x) = \sqrt{\frac{1}{x}+3}$

The differentiation of the product

Let $y = uv$ where u and v are functions of x

$\Rightarrow \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ is the product rule.

Proof

$\Rightarrow y + \delta y = (u + \delta u)(v + \delta v)$

Where a small increment δx in x produces increments δu in u, δv in v and δy in y.

$\Rightarrow y + \delta y = uv + u\delta v + v\delta u + \delta u\delta v$

And since $y = uv$

$\Rightarrow \delta y = u\delta v + v\delta u + \delta u\delta v$

Dividing through by δx

$\Rightarrow \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\delta v$

As $\delta x \to 0, \delta u, \delta v$ and δy also approach 0.

$\frac{\delta y}{\delta x} \to \frac{dy}{dx}, \frac{\delta u}{\delta x} \to \frac{du}{dx}, \frac{\delta v}{\delta x} \to \frac{dv}{dx}$

$\therefore \boldsymbol{\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}}$

This formula must be remembered, and this is perhaps most easily done in words

"To differentiate the product of two factors, differentiate the first factor, leaving the second one alone and then differentiate the second, leaving the first one alone"

So in case of three functions uvw the formula also applies.

$\boldsymbol{\frac{dy}{dx} = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}}$

Example 33

Find $\frac{dy}{dx}$ for each of the following;

(a) $y = x^2(x + 2)^3$
(b) $y = (x + 2)^2(1 - x^2)^4$
(c) $y = 7x^2\sqrt{x^2 - 1}$

Solution

(a) Let $u = x^2$; $\frac{du}{dx} = 2x$

Also $v = (x + 2)^3$; $\frac{dv}{dx} = 3(x + 2)^2(1)$

Using $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= (x + 2)^3(2x) + x^2(3(x + 2)^2)$

$= x(x + 2)^2(2 + 3x)$

(b) Let $u = (x + 2)^2$; $\frac{du}{dx} = 2(x + 2)$

Also $v = (1 - x^2)^4$; $\frac{dv}{dx} = 4(1 - x^2)^3(-2x)$

Using $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= (1 - x^2)^4(2(x + 2)) + (x + 2)^2\left(4(1 - x^2)^3(-2x)\right)$

$= 2(1 - x^2)^3(x + 2)\left((1 - x^2) - 4x\right)$

(c) Let $u = 7x^2$; $\frac{du}{dx} = 14x$

Also $v = \sqrt{x^2 - 1}$; $\frac{dv}{dx} = \frac{1}{2}(x^2 - 1)^{\frac{-1}{2}}(2x) = \frac{x}{\sqrt{x^2-1}}$

Using $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= 14x\sqrt{x^2 - 1} + 7x^2\left(\frac{x}{\sqrt{x^2-1}}\right)$

$= \frac{14x(x^2-1)+7x^3}{\sqrt{x^2-1}}$

$= \frac{14x^3-14x+7x^3}{\sqrt{x^2-1}}$

$= \frac{7x(3x^2-2)}{\sqrt{x^2-1}}$

Example 34

Show that if $y = (x - a)^2V$ where V is a polynomial in x then dy/dx is a polynomial with $(x - a)$ as a factor. Hence or otherwise find the values of the constants k and l for which $x^4 - 2x^3 + 5x^2 + kx + l$ has a factor $(x - 1)^2$ and solve the equation $x^4 - 2x^3 + 5x^2 + kx + l = 0$

Solution

$\frac{dy}{dx} = 2(x - a)V + (x - a)^2\frac{dV}{dx}$

$= (x - a)\left\{2V + (x - a)\frac{dV}{dx}\right\}$

If we let $x - a = 0; x = a$

$\Longrightarrow \frac{dy}{dx} = (a - a)\left\{2V + (a - a)\frac{dV}{dx}\right\} = 0$

$\therefore since \frac{dy}{dx} = 0, then\ x = a\ is\ a\ root\ and\ (x - a) is\ a factor\ of\ \frac{dy}{dx}$

Hence

If $x = 1$

$\Longrightarrow (1)^4 - 2(1)^3 + 5(1)^2 + k(1) + l = 0$

$\Longrightarrow k + l = -4$ ....(1)

Let $y = x^4 - 2x^3 + 5x^2 + kx + l$

$\Rightarrow \frac{dy}{dx} = 4x^3 - 6x^2 + 10x + k$

Using $x = 1$

$\Rightarrow 4(1)^3 - 6(1)^2 + 10(1) + k = 0; k = -8$

From (1) $l = -4 - (-8) = 4$

$\therefore k = -8 \text{ and } l = 4$

Exercise 8.5a

1. Differentiate the following with respect to x
   (a) $y = (1 + x^2)(1 - 2x^2)$
   (b) $y = x^2\sqrt{2 - x}$
   (c) $y = (1 - x^3)\sqrt{x + 1}$
   (d) $y = (x^2 + 1)(x + 1)^{\frac{1}{2}}$
   (e) $y = (x^2 + 1)^2(x - 1)^2$
2. Determine the turning point of the curve represented by $y = x^2(x - 3)$ at the point when $y = 0$.
3. If $f(x) = (x - a)^2 g(x)$, show that $(x - a)$ is a factor of $f'(x)$.
4. Given that $R = q\sqrt{1000 - q^2}$ find;
   (a) $\frac{dR}{dq}$
   (b) the value of q when R is a maximum.
5. Differentiate the following using the product rule;
   (a) $(x + 3)(x - 4)$ (b) $(x^2 - 5)(4x + 1)$ (c) $x^3(2 + x)^2$ (d) $5x^2(x^2 - x + 1)^4$

   (e) $x\sqrt{x + 1}$ (f) $(1 - 3x)\sqrt{2x + 5}$ (g) $\left(\sqrt{6 + x}\right)\sqrt{3 - 2x}$ (h) $(3x + 5)^2\sqrt{x - 2}$

The differentiation of the quotient

Suppose $y = \frac{u}{v}$ where u and v are given functions of x.

Then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ is the quotient rule

Proof

$y + \delta y = \frac{u + \delta u}{v + \delta v}$

Where a small increment $\delta x \text{ in } x$ produces increments δu in u, δv in v and δy in y.

$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$

$= \frac{v(u + \delta u) - u(v + \delta v)}{v(v + \delta v)}$

$= \frac{uv + v\delta u - uv - u\delta v}{v^2\left(1 + \frac{\delta v}{v}\right)}$

$= \frac{v\delta u - u\delta v}{v^2\left(1 + \frac{\delta v}{v}\right)}$

$\frac{\delta y}{\delta x} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2\left(1 + \frac{\delta v}{v}\right)}$

As $\delta x \to 0, \delta u, \delta v \text{ and } \delta y$ also approach 0.

$\frac{\delta y}{\delta x} \to \frac{dy}{dx}, \frac{\delta u}{\delta x} \to \frac{du}{dx}, \frac{\delta v}{\delta x} \to \frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 35

Differentiate the following with respect to x

(a) $y = \frac{3x+2}{2x+1}$

(b) $y = \frac{x}{(x^2+4)^3}$

(c) $y = \sqrt{\frac{x^3}{x^2-1}}$

Solution

(a) Let $u = 3x + 2$; $\frac{du}{dx} = 3$

Also $v = 2x + 1$; $\frac{dv}{dx} = 2$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(2x+1)(3)-(3x+2)(2)}{(2x+1)^2}$

$= \frac{6x+3-(6x+4)}{(2x+1)^2}$

$= \frac{-1}{(2x+1)^2}$

(b) Let $u = x$; $\frac{du}{dx} = 1$

Also $v = (x^2 + 4)^3$; $\frac{dv}{dx} = 3(x^2 + 4)^2(2x) = 6x(x^2 + 4)^2$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{(x^2+4)^3(1)-x\left(6x(x^2+4)^2\right)}{((x^2+4)^3)^2}$

$= \frac{(x^2+4)^2\{x^2+4-6x^2\}}{(x^2+4)^6}$

$= \frac{(x^2+4)^2(4-5x^2)}{(x^2+4)^6}$

$= \frac{(4-5x^2)}{(x^2+4)^4}$

(c) $y = \sqrt{\frac{x^3}{x^2-1}} = \frac{x^{\frac{3}{2}}}{(x^2-1)^{\frac{1}{2}}}$

Let $u = x^{\frac{3}{2}}$; $\frac{du}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

Also $v = (x^2 - 1)^{\frac{1}{2}}$; $\frac{dv}{dx} = \frac{1}{2}(x^2 - 1)^{\frac{-1}{2}}(2x) = \frac{x}{\sqrt{x^2-1}}$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$= \frac{\sqrt{x^2-1}\left(\frac{3}{2}x^{\frac{1}{2}}\right) - \left(x^{\frac{3}{2}}\right)\left(\frac{x}{\sqrt{x^2-1}}\right)}{(x^2-1)}$$

$$= \frac{x^{\frac{1}{2}}\left\{\frac{3}{2}(x^2-1) - x^2\right\}}{(x^2-1)\sqrt{x^2-1}}$$

$$= \frac{x^{\frac{1}{2}}(x^2-3)}{2(x^2-1)\sqrt{x^2-1}}$$

$$= \frac{x^{\frac{1}{2}}(x^2-3)}{2\sqrt{(x^2-1)^3}}$$

Example 36

If $y = x^2/\sqrt{(x+1)}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Solution

$$\frac{dy}{dx} = \frac{\sqrt{(x+1)}(2x) - x^2\left(\frac{1}{2}(x+1)^{\frac{-1}{2}}\right)}{(x+1)}$$

$$= \frac{\sqrt{(x+1)}(2x) - \frac{x^2}{2\sqrt{(x+1)}}}{(x+1)}$$

$$= \frac{4x(x+1) - x^2}{(x+1)}$$

$$= \frac{4x^2+4-x^2}{(x+1)}$$

$$\therefore \frac{dy}{dx} = \frac{3x^2+4}{(x+1)}$$

$$\Longrightarrow \frac{d^2y}{dx^2} = \frac{(x+1)(6x) - (3x^2+4)(1)}{(x+1)^2}$$

$$= \frac{6x^2+6x-3x^2-4}{(x+1)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{3x^2+6x-4}{(x+1)^2}$$

Exercise 8.5b

1. Differentiate the following with respect to x

   (a) $\frac{1}{(2x+3)}$ (b) $\frac{1}{\sqrt{3x+1}}$ (c) $\frac{1}{(2x-4)^{\frac{3}{4}}}$ (d) $\frac{-1}{(1+\sqrt{x})^2}$ (e) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^{\frac{1}{2}}$

2. Differentiate the following with respect to x

   (a) $\frac{x^2}{1+x^3}$ (b) $\frac{1}{(x^2+2x-4)^2}$ (c) $\frac{x^2-7x+4}{x^3}$ (d) $\frac{x^2-2}{2\sqrt{x}}$ (e) $\frac{x^2}{(3x-1)^3}$ (f) $\frac{3x+4}{\sqrt{2x^2+3x-2}}$

   (g) $\frac{(x^2-3)}{(x^2+2x-3)^{\frac{1}{2}}}$

3. Differentiate the following using the quotient rule:

(a) $\frac{x}{x-2}$ (b) $\frac{1+3x}{4+x}$ (c) $\frac{x^2}{2x-3}$ (d) $\frac{(3x-2)^2}{\sqrt{x}}$ (e) $\frac{3-\sqrt{x}}{(2+x)^2}$ (f) $\frac{(3x^2+2)^4}{\sqrt{2x-1}}$ (g) $\sqrt{\frac{x-2}{x+1}}$ (h) $\sqrt{\frac{x^2+1}{x^3-3}}$ (i) $\frac{\sqrt{x}+1}{\sqrt{x}-1}$ (j) $\frac{x^3\sqrt{4-x^2}}{5-\sqrt{x}}$ (k) $\frac{x(x-1)^3}{x-3}$

Applications of the rules:

These are cases where we see the use of the chain, product and the quotient rules

Example 37

Find the equation of the tangent to the curve $y = x^2(x+1)^4$ at the point P(1, 16)

Solution

$\frac{dy}{dx} = x^2(4(x+1)^3) + 2x(x+1)^4$

$= 2x(x+1)^3(2x + x + 1)$

$= 2x(x+1)^3(3x+1)$

At P when $x = 1$

$\Rightarrow \frac{dy}{dx} = 2(1)(1+1)^3(3(1)+1) = 64$

Gradient of the tangent is $m = 64$

Using $y = mx + c$

$\Rightarrow y = 64x + c$

At P (1, 16)

$\Rightarrow 16 = 64(1) + c; c = -48$

$\therefore y = 64x - 48$ is the equation of the tangent.

Example 38

Find the coordinates of the two points on the curve $y = \frac{x}{1+x}$ where the gradient is 1/9

Solution

$\frac{dy}{dx} = \frac{(1+x)(1)-(x)(1)}{(1+x)^2}$

$= \frac{1+x-x}{(1+x)^2}$

$= \frac{1}{(1+x)^2}$

But $\frac{dy}{dx} = \frac{1}{9}$

$\Rightarrow \frac{1}{9} = \frac{1}{(1+x)^2}$

$\Rightarrow 1 + 2x + x^2 = 9$

$\Rightarrow x^2 + 2x - 8 = 0$

$\Rightarrow x^2 + 4x - 2x - 8 = 0$

$\Rightarrow x(x + 4) - 2(x + 4) = 0$

$\Rightarrow (x - 2)(x + 4) = 0$

$\Rightarrow x = 2 \text{ and } x = -4$

When $x = 2, y = \frac{2}{1+2} = \frac{2}{3}$; $\left(\mathbf{2}, \frac{\mathbf{2}}{\mathbf{3}}\right)$

When $x = -4, y = \frac{-4}{1-4} = \frac{4}{3}$; $\left(\mathbf{-4}, \frac{\mathbf{4}}{\mathbf{3}}\right)$

$\therefore$ the required coordinates are $\left(\mathbf{2}, \frac{\mathbf{2}}{\mathbf{3}}\right)$ and $\left(\mathbf{-4}, \frac{\mathbf{4}}{\mathbf{3}}\right)$

Exercise 8.5c

1. The equation o a curve is given by $y = \frac{3x}{x+1}$
   Determine;
   (i) the slope of the curve at $x = 2$
   (ii) the coordinates of the points on the curve at which the tangents are parallel to the line $y = 3x + 7$
2. Given the curve $y = \frac{2x-1}{x^2+2}$, find the;
   (i) points where the curve cuts the axes.
   (ii) turning points of the curve and distinguish them.
3. (a) Find the coordinates of the stationary points on the curve $f(x) = \frac{x}{x^2+4}$
   (b) Show that $f''(x) = \frac{2x(x^2-12)}{(x^2+4)^2}$
   (c) Hence determine the nature of the turning points
   $Ans\left((a)(2, \frac{1}{4}), (-2, \frac{-1}{4})\ (c) max, \text{min}\ respectively\right)$
4. The tangent to the curve $y = 3x\sqrt{1 + 2x}$ at the point $(4, 36)$, meets the x-axis at P and the y-axis at Q. Calculate the area of the triangle OPQ. Where O is the origin. $Ans\left(9\frac{11}{13}\right)$
5. Show that there is just one point, P, on the curve $y = x(x - 1)^3$, where the gradient is 7. Find the coordinates of P. $Ans(2, 2)$

6. Given the curve $y = \frac{x^2}{2-x}$, show that $\frac{dy}{dx} = \frac{x(4-x)}{(2-x)^2}$. Hence find the coordinates of the two points on the curve where the gradient of the curve is zero. $Ans((0,0),(4,-8))$
7. Given that $y = x\sqrt{3+2x}$, show that $\frac{dy}{dx} = \frac{3(1+x)}{\sqrt{3+2x}}$. Hence find the point of the curve $y = x\sqrt{3+2x}$ where the gradient is zero. $Ans(-1,-1)$
8. The tangent to the curve $y = x^2 - 4$ at the point where $x = a$ meets the x-axis and y-axis at the points P and Q respectively.

(a) Show that the area of the triangle OPQ, where O is the origin, is given by $\frac{(a^2+4)^2}{4a}$.

(b) Hence find the minimum area of the triangle OPQ. $Ans\left(\frac{32\sqrt{3}}{9}\right)$

9. Given that $y = \frac{x^2+1}{x^2-2}$ find and simplify an expression for $\frac{dy}{dx}$. Hence determine the coordinates of the turning point on this curve, and determine its nature.

The parametric functions

In some cases, y is defined as a function of x by expressing both y and x in terms of a third variable known as a parameter. Such equations are called **parametric equations.**

For example, the pair of equations;

$x = t + 1$ ....(1)

$y = t^2$ ....(2) are parametric equations, with the parameter being t.

When the parameter t is eliminated, we form the Cartesian equation of the curve.

From (1) ; $t = x - 1$

Equation (2) becomes $y = (x-1)^2$

$\Rightarrow y = x^2 - 2x + 1$ is the Cartesian equation of the curve.

Example 39

Find the Cartesian equation for each of the following parametric forms

(a) $x = \sqrt{t}, y = 2t^2 - 1$

(b) $x = \frac{1}{t}, y = 3t - 1$

(c) $x = \frac{1}{2-t},\ y = \frac{3}{1+2t}$

Solution

(a) $t = x^2\ and\ y = 2(x^2)^2 - 1\ \therefore y = 2x^4 - 1$

(b) $t = \frac{1}{x}\ and\ y = 3\left(\frac{1}{x}\right) - 1 \quad \therefore y = \frac{3}{x} - 1$

(c) $\Rightarrow x(2-t)=1; 2x-xt=1; t=\frac{2x-1}{x}$

$\Rightarrow y=\frac{3}{1+2\left(\frac{2x-1}{x}\right)}=\frac{3x}{x+4x-2} \quad \therefore y=\frac{3x}{5x-2}$

Qn. Find the Cartesian equation represented by the parametric equations $x=\frac{t}{1+t}$ and $y=\frac{t^3}{1+t}$

Parametric differentiation

Here majorly the chain rule must be called into play in order to differentiate the parametric equations.

Example 40

Find $\frac{dy}{dx}$, given that $x=t+1$ and $y=t^2$

Solution

$\frac{dx}{dt}=1 \; and \; \frac{dy}{dt}=2t$

By the chain rule:

$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$ but $\frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}$

$\Rightarrow \frac{dy}{dx}=2t(1)=2t$

Example 41

Find $\frac{dy}{dx}$ in terms of the parameter for each of the following

(a) $y=3p^2+2p, x=1-2p$

(b) $y=(1+2t)^3, x=t^3$

Solution

(a) $\frac{dy}{dp}=6p+2 \; and \; \frac{dx}{dp}=-2$

Using the chain rule:

$\frac{dy}{dx}=\frac{dy}{dp}\times\frac{dp}{dx}$

$\Rightarrow \frac{dy}{dx}=(6p+2)\times\frac{1}{-2}=-(3p+1)$

(b) $\frac{dy}{dt}=3(1+2t)^2(2) \; and \; \frac{dx}{dt}=3t^2$

By the chain rule:

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\Longrightarrow \frac{dy}{dx} = (6(1+2t)^2)\left(\frac{1}{3t^2}\right) = \frac{2(1+2t)^2}{t^2}$

Example 42

Find the equation of the tangent to the curve given parametrically by $x = \frac{2}{t}$ and $y = 3t^2 - 1$ at the point (2, 2)

Solution

$\frac{dy}{dt} = 6t \; and \; \frac{dx}{dt} = -\frac{2}{t^2}$

By chain rule

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\Longrightarrow \frac{dy}{dx} = 6t\left(-\frac{t^2}{2}\right) = -3t^3$

At (2, 2), $\Longrightarrow 2 = \frac{2}{t}; \;\; t = 1$

Gradient of the tangent, $m = \frac{dy}{dx} = -3(1)^3 = -3$

The equation of the tangent becomes

$\Longrightarrow y = -3x + c$

At (2, 2), $2 = -3(2) + c; \;\; c = 8$

$\therefore y = -3x + 8$ is the equation of the tangent.

Example 43

Find the gradient of the following parametric equations $x = \frac{t}{1+t}$ and $y = \frac{t^3}{1+t}$. At the point (1/2 , 1/2)

Solution

$\frac{dx}{dt} = \frac{(1+t)(1)-(t)(1)}{(1+t)^2} = \frac{1}{(1+t)^2}$

$\frac{dy}{dt} = \frac{(1+t)(3t^2)-(t^3)(1)}{(1+t)^2} = \frac{3t^2+3t^3-t^3}{(1+t)^2} = \frac{3t^2+2t^3}{(1+t)^2}$

Using the chain rule;

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \left(\frac{3t^2+2t^3}{(1+t)^2}\right) \times \frac{(1+t)^2}{1}$

$= 3t^2 + 2t^3$

When $x = 1/2$; $\Longrightarrow \frac{1}{2} = \frac{t}{1+t}$

$\Longrightarrow 1 + t = 2t; \;\; t = 1$

Hence gradient $= 3(1)^2 + 2(1)^3 = 5$

NB. The **second derivative** of parametric equations is given by $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

Example 44

If $x = a(t^2 - 1)$ and $y = 2a(t + 1)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

Solution

$\frac{dx}{dt} = 2at$ and $\frac{dy}{dt} = 2a$

$\Longrightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= (2a) \times \left(\frac{1}{2at}\right) = \frac{1}{t}$

$\Longrightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$= \frac{d}{dt}\left(\frac{1}{t}\right) \times \left(\frac{1}{2at}\right)$

$= \left(-\frac{1}{t^2}\right) \times \left(\frac{1}{2at}\right)$

$= -\frac{1}{2at^3}$

Example 45

Given the following parametric equations $x = \frac{t}{1+t}$ and $y = \frac{t^3}{1+t}$ find $\frac{d^2y}{dx^2}$

Solution

$\frac{dx}{dt} = \frac{(1+t)(1)-(t)(1)}{(1+t)^2} = \frac{1}{(1+t)^2}$

$\frac{dy}{dt} = \frac{(1+t)(3t^2)-(t^3)(1)}{(1+t)^2} = \frac{3t^2+3t^3-t^3}{(1+t)^2} = \frac{3t^2+2t^3}{(1+t)^2}$

Using the chain rule;

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \left(\frac{3t^2+2t^3}{(1+t)^2}\right) \times \frac{(1+t)^2}{1}$

$= 3t^2 + 2t^3$

$\Longrightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

$= \frac{d}{dt}(3t^2 + 2t^3) \times \left(\frac{(1+t)^2}{1}\right)$

$= (6t + 6t^2)(1 + t)^2$

$= 6t(1 + t)(1 + t)^2$

$= 6t(1 + t)^3$

Example 46

Find and classify the stationary point on the curve $x = 4 - t^3$ and $y = t^2 - 2t$

Solution

Differentiating parametrically gives;

$\frac{dx}{dt} = -3t^2$ *and* $\frac{dy}{dt} = 2t - 2$

By chain rule;

$\frac{dy}{dx} = (2t-2)\left(-\frac{1}{3t^2}\right) = \frac{2-2t}{3t^2}$

At a stationary point, $\frac{dy}{dx} = 0$

$\Rightarrow \frac{2-2t}{3t^2} = 0; 2 - 2t = 0;\ \ t = 1$

To find the coordinates of the point corresponding to the value of $t = 1$, we substitute $t = 1$ into $x = 4 - t^2$ and $y = t^2 - 2t$

This gives $x = 3$ and $y = -1$

The coordinates of the stationary point are $(3, -1)$

Using $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \left(\frac{dt}{dx}\right)$

$= \frac{d}{dt}\left(\frac{2-2t}{3t^2}\right) \times \left(\frac{1}{-3t^2}\right)$

$= \left[\frac{3t^2(-2)-(2-2t)(6t)}{9t^4}\right] \times \left(\frac{1}{-3t^2}\right)$

$= \frac{-6t^2-12t+12t^2}{9t^4(-3t^2\ )}$

$= \frac{6t(t-2)}{-27t^6}$

At the stationary point, $t = 1$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{6(1)(1-2)}{-27(1)^6} = \frac{2}{9}$

Since $\frac{d^2y}{dx^2} > 0$, the stationary point $(3, -1)$ is a minimum.

Exercise 8.5d

1. Find the Cartesian equation from each of the following parametric equations;

   (a) $x = t + 3, y = t^2$ (b) $x = \frac{1}{1+2t},\ y = \frac{t}{1+2t}$ (c) $x = 2t - 1,\ y = 12t^2 - 5$

   (d) $x = 2\sqrt{t},\ y = 8t^2 + 5$ (e) $x = t, y = \frac{4}{t}$ (f) $x = \frac{1}{t+1},\ y = t(1+t)$

2. For the following curves, each of which is given in terms of a parameter t, find an expression for $\frac{dy}{dx}$ in terms of t.

   (a) $x = t^2, y = 4t - 1$ (b) $x = t^6,\ y = 6t^3 - 5$ (c) $x = 6t - 5, y = (2t-1)^3$ (d) $x = \frac{2}{\sqrt[3]{3t-4},}\ \ y = \sqrt[3]{6t+1}$ (e) $x = \frac{1}{t},\ y = t^2 + 4t - 3$

3. In each part of this question find the gradient of the stated curve at the point defined by the given value of the parameter t.

   (a) $x = t + 5,\ y = t^3 - 2t$ where $t = 2$

(b) $x = (t-2)^2,\ y = (3t+4)^3$ where $t = 0$

(c) $x = \sqrt{t-1},\ y = \frac{1}{t}$ where $t = 10$

(d) $x = \frac{1}{\sqrt{t-4}},\ y = \frac{3}{\sqrt{t^2-9}}$ where $t = 5$

4. If $x = t^3 + t^2, y = t^2 + t$ find $\frac{dy}{dx}$ in terms of t
5. Find the equation of the tangent to the curve $x = 3t^2, y = 7 + 12t$, at the point where $t = 2$
6. Find the equation of the tangent to the curve $x = t^2, y = 6t - 7$, at the point where $x = 1$
7. Find the equation of the tangent and the normal to the curve $x = 6t^2, y = t^3 - 4t$ at the point where $t = -1$
8. Find the equation of the tangents to the curve $x = \frac{4}{t},\ y = t^3 - 3t + 2$, at the point where the curve crosses the x-axis.
9. At which points are the tangents to the curve $x = 2t^2 - 3, y = t^3 - 6t^2 + 9t - 4$, parallel to the x-axis.
10. Find the equation of the normal to the curve $x = \frac{16}{t^2}, y = 2t - 3$ at the points where the curve crosses the line $x = 1$.
11. If $x = 1/(1+t^2)$ and $y = t/(1+t^2)$ find $\frac{dy}{dx}$ in terms of t
12. A curve is given by $x = t^2, y = 4t$. The tangent at the point where $t = 2$, meets the tangent at the point where $t = -1$, at the point P. Find the coordinates of P.
13. A curve is given parametrically as $x = 2t^2, y = 4t - t^4$. Find the equation of the normal to the curve at the point (2, 3).
14. A curve is given parametrically by $x = 3t\ and\ y = \frac{4}{t^2+1}$. Find the equation of the tangent to the curve at (2, 10).
15. A curve is represented by the parametric equations $x = 3t$ and $y = \frac{4}{t^2+1}$. Find the general equation of the tangent to the curve in terms of , y and t. Hence determine the equation of the tangent at the point (3, 2).
16. (a) Find the equation of the tangent to the curve $x = 8t + 1, y = 2t^2$ at the points (9, 2) and (-31, 32).

    (b) Show that these tangents are perpendicular, and find the coordinates of the point of intersection of the tangents.
17. (a) Given the curve $x = \frac{t}{1+t},\ y = \frac{t^2}{1+t}$, show that $\frac{dy}{dx} = t(t+2)$

    (b) Hence find the points of the curve where the gradient is 15.
18. (a) Given the curve $x = t^2(1-3t^2), y = 5t^3(4-t)$, show that $\frac{dy}{dx} = \frac{10t(t-3)}{6t^2-1}$

    (b) Hence find the values of the parameter t at the points on the curve where the gradient is 8.
19. If $x = at^2, y = 2at$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

20. A curve is given parametrically by $x = (t^2 - 1)^2, y = t^3$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

21. For the following curves of which is given in terms of a parameter t, find an expression for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(a) $x = 3t + 1, y = t^3 + 5$ (b) $x = \frac{1}{1+\sqrt{t}},\ y = \frac{1}{1-\sqrt{t}}$ (c) $x = \frac{1}{t+1},\ y = \frac{1}{t-1}$ (d) $x = \frac{t+1}{t-1}, y = \frac{1}{1-t}$ (e) $x = t^2(2t + 3), y = 3t^4 + 4t^3 + 1$

22. (a) For the curve $x = \frac{3t-1}{t},\ y = \frac{t^2+4}{t}$, show that;

(i) $\frac{dy}{dx} = t^2 - 4$ (ii) $\frac{d^2y}{dx^2} = 2t^3$

(b) Hence find and classify any stationary values on the curve.

Rates of change

- The derivative $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ is called the rate of change of y with respect to x.
- It shows how changes in y are related to changes in x. For example, if $\frac{dy}{dx} = 3$, then y is increasing 3 times as fast as x.
- Also this is an application of chain rule.

Example 47

A spherical balloon is inflated at a rate of 3cm^{3}s^{-1}. Find the rate of increase of the radius when this radius is 2 cm.

Solution

Let the balloon have radius r and volume V, then $V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dV}{dr} = 4\pi r^2$

We are given that $\frac{dV}{dt} = 3$ and we know that

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\Rightarrow 3 = 4\pi r^2 \times \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2}$

When $r = 2$

$\frac{dr}{dt} = \frac{3}{4\pi(2)^2} = \frac{3}{16\pi}$

$\therefore$ *the rate of increase of the radius when the radius is 2 cm is* $\frac{3}{16\pi} cms^{-1}$

Example 48

The volume of a cube is increasing at the rate of 18cm^{3}s^{-1}. Find he rate of increase of the length of a side when the volume is 125 cm^3

Solution

Volume V, of a cube is x^3 where $x\ cm$ is the length of the side.

$\Rightarrow V = x^3;\ \ 125 = x^3; x = 5\ cm$

$\Rightarrow \frac{dV}{dx} = 3x^2, \frac{dV}{dt} = 18cm^3s^{-1} required\ is \frac{dx}{dt}$

Using $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$\Rightarrow 18 = 3x^2 \frac{dx}{dt}$

When $x = 5\ cm$

$\Rightarrow \frac{dx}{dt} = \frac{6}{x^2} = \frac{6}{5^2} = 0.24cms^{-1}$

∴ the rate of increase of the length is 0.24 cms^{-1}.

Example 49

The side of a square is increasing at the rate of 5 cms^{-1}. Find the rate of increase of the area when the length of a side is 10 cm.

Solution

Let the side of a square be x cm

$\frac{dx}{dt} = 5cms^{-1},\ \ x = 10cm$

$\Rightarrow A = x^2\ and\ \frac{dA}{dx} = 2x$

Using $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$= 2x(5) = 10x$

When $x = 10\ cm$; $\frac{dA}{dt} = 10(10) = 100cm^2s^{-1}$

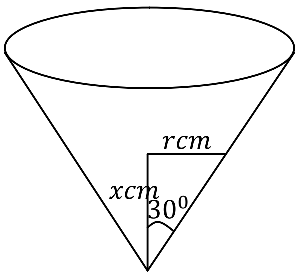
Example 50

A container in the shape of a hollow cone of semi- vertical angle 30^0 is held with its vertex pointing downwards. Water is poured into the cone at the rate of $5cm^3s^{-1}$

Solution

Let the depth of water in the cone be $x\ cm$

The radius, $r\ cm$, of the cross-section of water is given by.



$\Rightarrow \tan 30^0 = \frac{r}{x}$

$\therefore r = x \tan 30^0 = \frac{r}{\sqrt{3}}$

The volume, V cm^3, of the water in the cone is given by

$V = \frac{1}{3}\pi r^2 x = \frac{1}{3}\pi \left(\frac{x}{\sqrt{3}}\right)^2 x = \frac{1}{9}\pi x^3$

$\Rightarrow \frac{dV}{dx} = \frac{1}{3}\pi x^2$

We are given that $\frac{dV}{dt} = 5$ and we know that

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$\Rightarrow 5 = \frac{1}{3}\pi x^2 \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = \frac{15}{\pi x^2}$

When $x = 10$

$\frac{dx}{dt} = \frac{15}{\pi(10)^2} = \frac{3}{20\pi}$

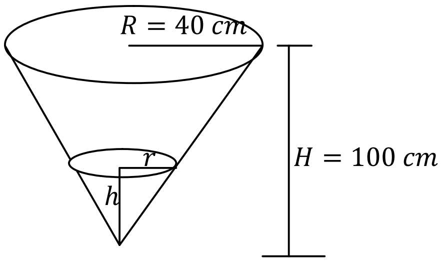
The rate of increase of the depth when the depth is 10 cm is, therefore $\frac{3}{20\pi}$ cms^{-1}

Example 51

A container is in the form of an inverted right circular cone. Its height is 100cm and base radius is 40cm. The container is full of water and has a small hole at the vertex. Water is flowing through the hole at a rate of $10cm^3s^{-1}$. Find the rate at which the water level in the container is falling when the height of water in the container is halved.

Solution

Let the height of water be $h\ cm$ and its radius $r\ cm$



By similarity $\frac{H}{R} = \frac{h}{r}$

$\Rightarrow \frac{100}{40} = \frac{h}{r};\ \ r = \frac{2}{5}h$

$And\ \frac{dV}{dt} = 10\text{cm}^3\text{s}^{-1}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{75}\pi h^3$

$\frac{dV}{dh} = \frac{13}{75}\pi h^2$

Using $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$\Rightarrow 10 = \frac{12}{75}\pi h^2 \frac{dh}{dt};\ \ \frac{dh}{dt} = \frac{750}{12\pi h^2}$

When $h = 50\ cm$

$\frac{dh}{dt} = \frac{750}{12\pi(50)^2} = \frac{1}{40\pi} cms^{-1}$

The rate of falling of the water level when the depth is 50 cm is, therefore $\frac{1}{40\pi}$ cms^{-1}

Exercise 8.5e

1. A spherical balloon is inflated at the rate of 2cm^3 every second. What is the rate of growth of the radius?
2. The sides of a cube are increasing at a rate of 6 cm/s. Find the rate of increase of the volume when the length of a side is 9 cm.
3. The side of a square is increasing at the rate of 3cms^{-1}. Find the rate of increase of the area of a square of side is 4 cm. $\boldsymbol{Ans(24cm^2s^{-1})}$
4. The surface area of a sphere is increasing at a rate of 2 cm^{2}s^{-1}. Find the rate of increase of the radius when the surface area is $(100\pi)cm^2$. $Ans\left(\frac{2\pi}{5}cms^{-1}\right)$
5. The diameter of an expanding smoke ring at time t is proportional to t^2. If the diameter is 6 cm after 6 seconds. At what rate is it then changing? $Ans(2\ cm/s)$
6. Air is escaping from a spherical ball at a rate of 0.30m^3 per minute. Find how fast the radius is decreasing when radius is 0.3m.
7. When sand is poured on level ground, the heap of sand takes on a conical shape with semi-vertical angle of 45^0. Calculate the volume of the sand when the height is $h\ m$. If the sand is poured at a constant rate of $1cm^3min^{-1}$. Calculate the rate at which the height of the sand is increasing when the height of the heap is 0.5m.
8. An inverted cone with a vertical angle of 60^0 is collecting water leaking from a tap at a rate of $0.2cm^3s^{-1}$. If the height of the water collected in the cone is 10cm, find the rate at which the surface area of water is increasing.
9. A container in the shape of a hollow cone of vertical angle 90^0 is held with its vertex pointing downwards. Water drips into the container at a rate of 3 cm^3 per minute. Find the rate at which the depth of water in the cone is increasing when this depth is 2 cm. $Ans(\frac{3}{4\pi}cm\ min^{-1})$
10. After t seconds the area A sq.cm of an ink stain is given by $A = t + t^3$. Find the rate at which the area is increasing after 2 seconds.
11. A spherical balloon is blown such that its volume increases at a constant rate of 8.0cm^{3}s^{-1}. Find the rate of increase of its radius when the volume of the balloon is 200cm^3.
12. The surface area of a spherical ball is increasing at a rate of 0.8m^{2}s^{-1} when its radius is 0.65m. Find the rate of change of its volume at that instant.
13. Oil is dropping onto a surface of water at the rate of $\frac{1}{10}\pi cm^3s^{-1}$. It forms a cylindrical film which has a height of 0.1cm. Find the rate at which the radius of the film is increasing when the radius of the film is 5 cm.
14. A cylindrical can is made in such away that the sum of its height and the circumference of its base is 45cm. Find the radius of the base of the cylinder when the volume of the can is maximum.
15. The surface area of a cube is increasing at a constant rate of 10cm^{3}s^{-1}. Find the rate of the volume when the side of the cube is 12cm long.

16. The area of a circular lamina increases at a rate of $10cm^2s^{-1}$. Find the rate of increase of the circumference at the instant when the radius is 50cm.
17. The inside of a glass is in the shape of an inverted cone of depth 8 cm and radius 4 cm. Wine is poured into the glass at the rate of 4 cm^3s^{-1}. Find the rate at which the depth of the wine in the glass is increasing when this depth is 6 cm. $Ans\left(\frac{4}{9\pi}cms^{-1}\right)$
18. A container in the shape of a hollow cone of depth 12cm and radius 6 cm is held with its vertex pointing downwards. Water is poured into the container at a rate of $2cm^3s^{-1}$. Find the rate at which the depth of water in the cone is increasing when this depth is $3/4\ cm$. $Ans\left(\frac{128}{9\pi}cms^{-1}\right)$
19. A boy is inflating a spherical balloon such that its radius is increasing at the rate of $x\ m/s$. Find the rate at which;
    (i) the surface area
    (ii) the volume of the balloon is increasing when the radius is 0.9m.
20. The area of an equilateral triangle increases at a rate of $24cm^2s^{-1}$. Find the rate of increase of the side of the triangle when the side is 8cm.
21. A container in the shape of a right circular cone of height 10 cm and base radius 1 cm is catching the drips from a tap leaking at the rate of $0.1cm^3/s$. Find the rate at which the surface area of water is increasing when the water is half way up the cone.
22. A hollow cone of base radius 10 cm and height 10 cm with its vertex downwards. The cone is initially empty when water is poured into it at the rate of $(4\pi)cm^3s^{-1}$. Find the rate of increase in the depth of water in the cone 18 seconds after pouring has commenced. $Ans\left(\frac{1}{9}cms^{-1}\right)$

8.6 Implicit Functions

Functions which cannot be expressed in the form $y = f(x)$ are called implicit functions. E.g $x^2 + 3xy + y^2 = 4$.

Implicit differentiation:

Example 52

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 2$

Solution

Differentiating each term with respect to x gives;

$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(2)$ ....(1)

By chain rule;

$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$

Therefore equation (1) becomes

$2x + 2y\frac{dy}{dx} = 0$

Rearranging for $\frac{dy}{dx}$ gives;

$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$

Example 53

Find $\frac{dy}{dx}$ for each of the following functions.

(a) $x^2 - 6y^3 + y = 0$

(b) $x^2y = 5x + 2$

(c) $(x + y)^2 - 7x^2 = 0$

(d) $\frac{x^3}{x+y} = 2$

Solution

(a) Differentiating each term of $x^2 - 6y^3 + y = 0$ with respect to x gives

$2x - 18y^2\frac{dy}{dx} + \frac{dy}{dx} = 0$

Rearranging for $\frac{dy}{dx}$ gives

$\Rightarrow \frac{dy}{dx}(1 - 18y^2) = -2x$

$\therefore \frac{dy}{dx} = \frac{-2x}{1-18y^2} = \frac{2x}{18y^2-1}$

(b) This requires the use of the product rule;

$\Rightarrow x^2\frac{dy}{dx} + y(2x) = 5$

$\therefore \frac{dy}{dx} = \frac{5-2xy}{x^2}$

(c) Differentiating each term of $(x + y)^2 - 7x^2 = 0$ with respect to x gives

$5(x + y)^4\left(1 + \frac{dy}{dx}\right) - 14x = 0$

$\Rightarrow 1 + \frac{dy}{dx} = \frac{14x}{5(x+y)^4}$

$\therefore \frac{dy}{dx} = \frac{14x}{5(x+y)^4} - 1$

(e) We have $\frac{x^3}{x+y} = 2$. Multiply through by $(x + y)$ gives

$x^3 = 2x + 2y$

Differentiating each term with respect to x gives

$3x^2 = 2 + 2\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{3x^2-2}{2}$

Example 54

Differentiate the following with respect to x

(a) $x^2 + y^2 - 6xy + 3x - 5y = 0$

(b) $x^2 - 3yx + 2y^2 - 2x = 4$ at $(1, -1)$

Solution

(a) Differentiating each term with respect to x gives;

$2x + 2y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx} + 3 - 5\frac{dy}{dx} = 0$

$\Rightarrow (2y - 6x - 5)\frac{dy}{dx} = 6y - 2x - 3$

$\Rightarrow \frac{dy}{dx} = \frac{6y-2x-3}{2y-6x-5}$

$at\ (1,-1); x = 1\ and\ y = -1$

$\therefore \frac{dy}{dx} = \frac{6(-1)-2(1)-3}{2(-1)-6(1)-5} = \frac{-11}{-13} = \frac{11}{13}$

(b) Differentiating each term with respect to x gives;

$2x - 3y - 3x\frac{dy}{dx} + 4y\frac{dy}{dx} - 2 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{3y-2x+2}{4y-3x}$

$at\ (1,-1)x = 1\ and\ y = -1$

$\therefore \frac{dy}{dx} = \frac{3(-1)-2(1)+2}{4(-1)-3(1)} = \frac{-3}{-7} = \frac{3}{7}$

Example 55

Find the gradient of the curve $x^2 + 2xy - 2y^2 + x = 2$ at the point $(-4, 1)$

Solution

Differentiating each term with respect to x gives;

$2x + 2y + 2x\frac{dy}{dx} - 4y\frac{dy}{dx} + 1 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-(2x+2y+1)}{(2x-4y)}$

At $(-4, 1); x = -4, y = 1$

$\frac{dy}{dx} = \frac{-(2(-4)+2(1)+1)}{2(-4)-4(1)} = \frac{5}{-12} = -\frac{5}{12}$

$\therefore$ the gradient of the curve is $-\frac{5}{12}$

Example 56

Find the equation of the normal to the curve $x^2 + 3xy + 2y^2 = 10$ at the point where $x = -1$

Solution

To find the y coordinates of the points on the curve where $x = -1$, substitute $x = -1$ into $x^2 + 3xy + 2y^2 = 10$

$\Rightarrow (-1)^2 + 3(-1)y + 2y^2 = 10$

$\Rightarrow 2y^2 - 3y + 1 = 10$

$\Rightarrow 2y^2 - 3y - 9 = 0$

$\Rightarrow (2y + 3)(y - 3) = 0$

Solving gives; $y = -\frac{3}{2}\ and\ y = 3$

The points on the curve at which $x = -1$ are $P\left(-1, -\frac{3}{2}\right)\ and\ Q(-1, 3)$

To find the equation of the normal at each point, we need the gradient of the curve at each point.

Since $x^2 + 3xy + 2y^2 = 10$, differentiating implicitly gives;

$2x + \left(3x\frac{dy}{dx} + 3y\right) + 4y\frac{dy}{dx} = 0$

$\Longrightarrow \frac{dy}{dx} = -\frac{(2x+3y)}{(3x+4y)}$

At the point $P\left(-1, -\frac{3}{2}\right)$

$\frac{dy}{dx} = -\frac{\left(2(-1)+3\left(-\frac{3}{2}\right)\right)}{\left(3(-1)+4\left(-\frac{3}{2}\right)\right)} = -\frac{13}{18}$

At the point $Q(-1, 3)$

$\frac{dy}{dx} = -\frac{(2(-1)+3(3))}{(3(-1)+4(3))} = -\frac{7}{9}$

The normal to the curve at point P has a gradient of $\frac{-1}{\left(-\frac{13}{18}\right)} = \frac{18}{13}$

So its equation has the form $y = \frac{18}{13}x + c_1$

Since the normal passes through $P\left(-1, -\frac{3}{2}\right)$, we have;

$-\frac{3}{2} = \frac{18}{13}(-1) + c_1;\ \ c_1 = -\frac{3}{26}$

$\therefore$ the equation of the bormal at P is $y = \frac{18}{13}x - \frac{3}{26}\ or\ 26y = 36x - 3$

The normal to the curve at point Q has a gradient of $\frac{-1}{\left(-\frac{7}{9}\right)} = \frac{9}{7}$

So its equation has the form $y = \frac{9}{7}x + c_2$

Since the normal passes through $Q(-1,\ 3)$, we have;

$3 = \frac{9}{7}(-1) + c_2;\ \ c_2 = \frac{30}{7}$

$\therefore$ the equation of the bormal at P is $y = \frac{9}{7}x + \frac{30}{7}\ or\ 7y = 9x + 30$

Second derivative:

Example 57

Find $\frac{d^2y}{dx^2}$ given that $x^2 + y^2 = 2$

Solution

$\Longrightarrow 2x + 2y\frac{dy}{dx} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$

$\Longrightarrow 2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0 \ldots\ldots\ldots\ldots(2)$

From (1); $\frac{dy}{dx} = -\frac{x}{y}$

Equation (2) becomes;

$\Longrightarrow 2 + 2\left(-\frac{x}{y}\right)^2 + 2y\frac{d^2y}{dx^2} = 0$

$\therefore \frac{d^2y}{dx^2} = -\left(\frac{y^2+x^2}{y^3}\right)$

Since $x^2 + y^2 = 2$, we have

$\therefore \frac{d^2y}{dx^2} = -\left(\frac{2}{y^3}\right)$

Example 58

Find and classify the stationary points on the curve $x^2 + xy + y^2 = 27$

Solution

At a stationary point, $\frac{dy}{dx} = y' = 0$. Differentiating $x^2 + xy + y^2 = 27$

$\Rightarrow 2x + (xy' + y) + 2yy' = 0$ ....(1)

Rearranging for y' gives;

$2yy' + xy' = -2x - y$

$\Rightarrow y'(2y + x) = -(2x + y)$

$\therefore y' = -\frac{(2x+y)}{(2y+x)}$

When $y' = 0$, we have

$-\frac{(2x+y)}{(2y+x)} = 0$

$\Rightarrow 2x + y = 0; y = -2x$ ....(2)

Substituting $y = -2x$ into $x^2 + xy + y^2 = 27$ gives

$x^2 + x(-2x) + (-2x)^2 = 27$

$\Rightarrow 3x^2 = 27; x = \pm 3$

Using equation (2), when $x = 3, y = -6$, when $x = -3, y = 6$. Therefore, the stationary points are $(3, -6) and\ (-3, 6)$

Differentiating equation (1) implicitly gives;

$2 + xy'' + y' + y' + 2yy'' + 2(y')^2 = 0$

But $y' = 0$ at stationary point.

$\Rightarrow 2 + xy'' + 2yy'' = 0$

To determine the nature of the turning points, we will examine the sign of y'' at each point.

At $(3, -6); 2 + 3y'' + 2(-6)y'' = 0;\ \ y'' = \frac{2}{9} > 0$

Since $y'' > 0$, the stationary point $(3, -6)$ is a minimum.

At $(-3,6); 2 - 3y'' + 2(6)y'' = 0;\ \ y'' = -\frac{2}{9} < 0$

Since $y'' < 0$, the stationary point $(-3, 6)$ is a maximum.

Example 59

Find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x^2 + y^2 - 6xy + 3x - 5y = 0$ at $(1, -1)$

Solution

Differentiating each term with respect to x gives

$2x + 2y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx} + 3 - 5\frac{dy}{dx} = 0$ ....(1)

$at(1, -1)$

$\Rightarrow 2(1) + 2(-1)\frac{dy}{dx} - 6(-1) - 6(1)\frac{dy}{dx} + 3 - 5\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{13}$

Differentiating equation (1) with respect to x

$\Rightarrow 2 + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 - 6\frac{dy}{dx} - 6\frac{dy}{dx} - 6x\frac{d^2y}{dx^2} - 5\frac{d^2y}{dx^2} = 0$

$\Rightarrow 2 + (2y - 6x - 5)\frac{d^2y}{dx^2} + (-12)\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 = 0$

$\Rightarrow 2+(2(-1)-6(1)-5)\frac{d^2y}{dx^2}-12\left(-\frac{1}{13}\right)+2\left(-\frac{1}{13}\right)^2=0$

$\therefore \frac{d^2y}{dx^2}=-\frac{496}{2197}$

Example 60

If $y=\sqrt{5x^2+3}$ show that $y\frac{d^2y}{dx^2}+\left[\frac{dy}{dx}\right]^2=5$

Solution

Squaring both sides gives;

$\Rightarrow y^2=5x^2+3$

Differentiating with respect to x gives

$\Rightarrow 2y\frac{dy}{dx}=10x$; $y\frac{dy}{dx}=5x$ on dividing through by 2

Differentiating for the second time we get

$\Rightarrow y\frac{d^2y}{dx^2}+\left[\frac{dy}{dx}\right]^2=5$

Exercise 8.6

1. For each of the following curves express $\frac{dy}{dx}$ in terms of x and y

(a) $x^2-y^2=4$ (b) $3xy-y^2=7$ (c) $x^2y+xy^2=2$ (d) $2x-y^3=3xy$ (e) $x^4-yx^2=6x$ (f) $x^2(x-3y)=4$ (g) $\frac{x^2}{x+y}=2$ (h) $\frac{y}{x^2-7y^3}=x^5$ (i) $x^6-5xy^3=9xy$

2. In each of this question find the gradient of the stated curve at the point specified.

(a) $xy^2-6y=8\ at\ (2,-1)$

(b) $x^4-y^3=3\ at\ (1,-1)$

(c) $3y^4-7xy^2-12y=5\ at\ (-2,1)$

(d) $(x+y)^2-4x+y+10=0\ at\ (2,-3)$

(e) $\frac{x^2}{x-y}=8\ at\ (4,\ 2)$

3. Find the equation of the tangent to the curve $xy^2+x^2y=6$ at the point (1, -3)
4. Find the equation of the tangent to the curve $xy^2-x^2y=12$ at the point where $y=3$
5. Find the equation of the normal to the curve $x^2+3x-2y^2=4$ at the point where the curve crosses the x-axis.
6. Find and classify the stationary values on each of the following curves;

(a) $x^2+y^2-4x+6y+12=0$

(b) $3x^2+y^2-6x+4y+6=0$

(c) $3x^2+x^2y+y=2$

(d) $xy+y-x^2=8$

(e) $2xy+y^2-x^2=0$

(f) $4xy^3=1+4x^2y^2$

7. Given that $x^n+y^n=1$, show that $\frac{d^2y}{dx^2}=-\frac{(n-1)x^{n-2}}{y^{2n-1}}$

8. Given that $x^2 + xy + y^2 - 3x - y = 3$

(a) Show that $\frac{dy}{dx} = \frac{3-2x-y}{x+2y-1}$

(b) Find, and classify, the maximum and minimum values of y

(c) Determine the coordinates of the points on the curve where the tangents to the curve are parallel to the y-axis.

8.7 Small Changes

This deals with the approximations of various variables;

Method I

If δx is small then; $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$ or $\delta y \approx \frac{dy}{dx} \times \delta x$

Thus if y is given a function of x we can determine the change in y corresponding to some given small change in x.

Method II

Since $f'(x) = \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\}$, it follows that when h is small

$f'(x) \approx \frac{f(x+h)-f(x)}{h}$

$\therefore \boldsymbol{f(x+h) \approx f(x) + hf'(x)}$

Example 61

If $y = 2x^2 - 3x$ find the approximate change in y when x increases from increases from 6 to 6.02

Solution

Let $y = 2x^2 - 3x$; $\frac{dy}{dx} = 4x - 3$ and $\delta x = 6.02 - 6 = 0.02$

Using $\delta y \approx \frac{dy}{dx} \times \delta x$

$\approx (4x - 3) \times 0.02$

$\approx (4(6) - 3) \times 0.02$

≈ 0.42

$\therefore$ the approximate change in y is 0.42

Example 62

Find an approximation of $\sqrt[3]{220}$

Solution

Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$;

$\Rightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

Using $f(x+h) \approx f(x) + hf'(x)$

$\Rightarrow \sqrt[3]{(x+h)} \approx \sqrt[3]{x} + \frac{h}{3\sqrt[3]{x^2}}$
$Let x = 216, h = 4$
$\Rightarrow \sqrt[3]{220} \approx \sqrt[3]{216} + \frac{4}{3\sqrt[3]{(216)^2}}$
$\approx 6 + \frac{4}{3\times 36}$
≈ 6.04
$\therefore \sqrt[3]{220} \approx 6.04$

Example 63

In calculating the area of a circle it is known that an error of $\pm 3\%$ could have been made in the measurement of the radius. Find the possible percentage error in the area.

Solution

$\Rightarrow \frac{\delta r}{r} = \frac{\pm 3}{100}$ and we require $\frac{\delta A}{A}$
For small changes $\frac{\delta A}{\delta r} \approx \frac{dA}{dr}$; but $A = \pi r^2$ and $\frac{dA}{dr} = 2\pi r$
$\Rightarrow \delta A \approx \frac{dA}{dr} \times \delta r$; $\delta r = \frac{\pm 3}{100} r$
$\Rightarrow \delta A \approx 2\pi r \times \frac{\pm 3}{100} r$
$\Rightarrow \frac{\delta A}{A} \approx \frac{2\pi r}{\pi r^2} \times \frac{\pm 3}{100} r$
$\Rightarrow \frac{\delta A}{A} \approx \frac{\pm 6}{100}$
$\therefore$ the percentage error in the area is $\pm 6\%$

Example 64

The time period, T, of a pendulum of length l is given by $T = 2\pi\sqrt{\frac{l}{g}}$ where π and g are constants. Find the approximate percentage increase in T when the length of the pendulum increases by 4%.

Solution

$\Rightarrow T^2 = \frac{4\pi l}{g}$
By implicit differentiation
$\Rightarrow 2T\frac{dT}{dl} = \frac{4\pi}{g}$; $\frac{dT}{dl} = \frac{2\pi}{gT}$
But $\frac{\delta l}{l} \approx \frac{4}{100}$; $\delta l = \frac{4}{100} l$
Using $\frac{\delta T}{\delta l} \approx \frac{dT}{dl}$
$\Rightarrow \delta T \approx \frac{2\pi}{gT} \times \frac{4}{100} l$
$\Rightarrow \frac{\delta T}{T} \approx \frac{2\pi}{gT^2} \times \frac{4}{100} l$
$\approx \frac{2\pi}{g\left(\frac{4\pi l}{g}\right)} \times \frac{4}{100} l$
$\approx \frac{2}{100}$
$\therefore$ percentage increase in T is 2%

Exercise 8.7

1. Find the approximate value of $\sqrt{25.01}$
2. Find an approximate value for $\sqrt{16.08}$ $Ans(4.01)$
3. Find the approximate value of $\sqrt[3]{65}$ correct to 3 significant figures
4. Find an approximate value of $\sqrt[3]{64.96}$
5. If $y = x^2 + 2x$ find the approximate increase in y when x increases from 9 to 9.01
6. If $y = 5x^4$ and x increases by 2% of its original value. Find the corresponding percentage increase in y.
7. Given that $y = x^{-\frac{1}{3}}$ use the calculus to determine an approximate value for $\frac{1}{\sqrt[3]{0.9}}$
8. Find an approximate value of $(5.02)^3$
9. The equation of a curve is $y = 2x^3 - 7x^2 + 15$. Write down an expression for $\frac{dy}{dx}$ and hence find;
   (a) The equation of the tangent to the curve at $(2, 3)$
   (b) The approximate change in y as x increases from 2 to 2.033, stating whether this is an increase or a decrease.
10. In measuring the volume of an sphere a 15% error was made. Find the percentage error that was made in measuring the radius.
11. Use calculus of increment to find $\sqrt{96}$ to one decimal place.
12. The surface area of a sphere increases by 2%. Find the corresponding percentage increase in radius of the sphere.

8.8 Partial fractions

We can use partial fractions to simplify expressions before differentiation

Example 65

Express $y = \frac{9x^2+34x+14}{(x+2)(x^2-x-12)}$ in partial fractions and hence find $\frac{dy}{dx}$ when $x = 0$

Solution

$x^2 - x - 12 = (x + 3)(x - 4)$

Therefore

$$\frac{9x^2+34x+14}{(x+2)(x^2-x-12)} \equiv \frac{9x^2+34x+14}{(x+2)(x+3)(x-4)}$$

Let $\frac{9x^2+34x+14}{(x+2)(x+3)(x-4)} \equiv \frac{A}{(x+2)} + \frac{B}{(x+3)} + \frac{C}{(x-4)}$

Putting $x = -2\ in\ \frac{9x^2+34x+14}{(x+3)(x-4)}\ gives\ A = \frac{9(-2)^2+34(-2)+14}{((-2)+3)((-2)-4)} = 3$

Putting $x = -3\ in\ \frac{9x^2+34x+14}{(x+2)(x-4)}\ gives\ B = \frac{9(-3)^2+34(-3)+14}{((-3)+3)((-3)-4)} = -1$

Putting $x = 4\ in\ \frac{9x^2+34x+14}{(x+2)(x+3)}\ gives\ C = \frac{9(4)^2+34(4)+14}{((4)+3)((4)-4)} = 7$

$$\therefore y = \frac{9x^2+34x+14}{(x+2)(x^2-x-12)} = \frac{3}{(x+2)} - \frac{1}{(x+3)} + \frac{7}{(x-4)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\frac{3}{(x+2)}\right] - \frac{d}{dx}\left[\frac{1}{(x+3)}\right] + \frac{d}{dx}\left[\frac{7}{(x-4)}\right]$$

$$= \frac{-3}{(x+2)^2} + \frac{1}{(x+3)^2} - \frac{7}{(x-4)^2}$$

When $x = 0$

$\Longrightarrow \frac{dy}{dx} = \frac{-3}{(0+2)^2} + \frac{1}{(0+3)^2} - \frac{7}{(0-4)^2}$

$= -\frac{3}{4} + \frac{1}{9} - \frac{7}{16}$

$= -\frac{155}{144}$

Exercise 8.8

1. Express $y = \frac{x-1}{(x+1)^2(x+2)}$ into partial fractions and hence find $\frac{dy}{dx}$
2. Express $y = \frac{2(x^3+x^2+1)}{(x-1)^2(x^2+1)}$ into partial fractions and hence find $\frac{dy}{dx}$ when $x = 0$
3. Express $\frac{13x+16}{(x-3)(3x+2)}$ in partial fractions. Hence find the value of $\frac{d}{dx}\left[\frac{13x+16}{(x-3)(3x+2)}\right]$ when $x = 2$
4. Express $y = \frac{2x^2-6x-1}{(x-1)(x-4)}$ in partial fractions, and hence show that $\frac{dy}{dx} = \frac{13}{4}$ when $x = 0$
5. Express $f(x) = \frac{3x}{(x-1)(x-4)}$ in partial fractions. Use your result to write down an expression for $f'(x)$ and show that $f''(x) = \frac{-2}{(x-1)^3} + \frac{8}{(x-4)^3}$

SENIOR FIVE TERM THREE

CHAPTER 9

INTEGRATION I

9.1 Indefinite Integrals

The reverse of differentiation

- In the general case, given $\frac{dy}{dx} = ax^n$, then $y = \frac{ax^{n+1}}{n+1} + c$ provided $n \neq -1$
- Thus the statement can be verified by differentiation;

  If $y = \frac{ax^{n+1}}{n+1} + c$; $\frac{dy}{dx} = \frac{a(n+1)x^{n+1-1}}{(n+1)} = ax^n$ as required

- The process of reverse differentiation is actually called **integration.** ax^n is called the **integrand** and the result is called the **integral.**
- We can now state the rule for integrating ax^n as $''increase, the\ power\ of\ x\ by\ one, amd\ divide\ by\ the\ new\ power''$
- Integrating a function, we do it on every part of the function.
- Using the integral sign $\int$ we write this as $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ for $n \neq -1$, c the constant of integration. With the $''dx''$ signifying that the integration is carried out with respect to the variable x.

- $\int g(x)\,dx$ is read as 'the integral of $g(x)$ with respect to x ' and we use the extended S symbol because integration can be considered as a summation.

Example 1

Integrate the following with respect to x.

(a) 2 (b) m (c) $3x$ (d) $3x^4$ (e) $3+2x$ (f) $x-x^2$ (g) $ax+b$

Solution

(a) $\int 2\,dx = \int 2x^0 dx = \frac{2x^{0+1}}{0+1} + c = 2x + c$

(b) $\int m dx = mx + c$

(c) $\int 3x\,dx = \frac{3x^2}{2} + c$

(d) $\int (3+2x)\,dx = 3x + x^2 + c$

(e) $\int (x - x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3} + c$

(f) $\int (ax+b)\,dx = \frac{ax^2}{2} + bx + c$ where c is an arbitrary constant.

Example 2

Find an expression for y if $\frac{dy}{dx}$ is given by

(a) $3x^2$ (b) $2x^5$ (c) $6x^2 - 4x + 2$ (d) $x^2(2x+1)$

Solution

(a) $\frac{dy}{dx} = 3x^2;\ \therefore y = \frac{3x^3}{3} + c = x^3 + c$

(b) $\frac{dy}{dx} = 2x^5;\ \therefore y = \frac{2x^6}{6} + c = \frac{x^6}{3} + c$

(c) $\frac{dy}{dx} = 6x^2 - 4x + 2;\ \therefore y = \frac{6x^3}{3} - \frac{4x^2}{2} + 2x + c = 2x^3 - 2x^2 + 2x + c$

(d) $\frac{dy}{dx} = x^2(2x+1) = 2x^3 + x^2;\ \therefore y = \frac{2x^4}{4} + \frac{x^3}{3} + c = \frac{x^4}{2} + \frac{x^3}{3} + c$

Example 3

Find (a) $\int 4x^2\,dx$ (b) $\int \frac{6}{x^3} dx$ (c) $\int \left(5 - x^2 + \frac{18}{x^4}\right) dx$ (d) $\int \frac{(y^2+2)(y^2-3)}{y^2} dy$

Solution

(a) $\int 4x^2\,dx = \frac{4x^3}{3} + c$

(b) $\int \frac{6}{x^3} dx = \int 6x^{-3}\,dx = \frac{6x^{-2}}{-2} + c = -\frac{3}{x^2} + c$

(c) $\int \left(5 - x^2 + \frac{18}{x^4}\right) dx = \int (5 - x^2 + 18x^{-4})\,dx$

$= 5x - \frac{x^3}{3} + \frac{18x^{-3}}{-3} + c$

$= 5x - \frac{x^3}{3} - \frac{6}{x^3} + c$

(d) $\int \frac{(y^2+2)(y^2-3)}{y^2}\, dy = \int \left(\frac{y^4-y^2-6}{y^2}\right) dy$

$=\int \left(\frac{y^4}{y^2} - \frac{y^2}{y^2} - \frac{6}{y^2}\right) dy$

$=\int (y^2 - 1 - 6y^{-2})\, dy$

$=\frac{y^3}{3} - y - \frac{6y^{-1}}{-1} + c$

$=\frac{y^3}{3} - 6 + \frac{6}{y} + c$

Since there is an arbitrary constant c in each of these solutions, we say that these are **indefinite integrals**

9.2 Definite Integral

- The integral $\int_a^b f(x)\, dx$ is known as a definite integral because of the limits of integration i.e. $x = a\ \ and\ x = b$ are known. Where $a\ is\ the\ lower\ limit$ and $b\ is\ the\ upper\ limit$.
- Suppose $\int_a^b f(x)\, dx = [F(x) + C]_a^b$

  $= \{(F(b) + C) - (F(a) + C)\}$

  $=F(b) - F(a)$
- Therefore we write this as $\int_{x=a}^{x=b} f(x)\, dx = F(b) - F(a)$

Example 4

Evaluate the following integrals

(a) $\int_{-1}^{1}(2x - 3)\, dx$ (b) $\int_{1/4}^{1/2}\left(\frac{1}{x^3}\right) dx$ (c) $\int_{-1}^{1}(3x^2 - 5)\, dx$ (d) $\int_2^8 \left(x - \frac{3}{\sqrt{x}}\right) dx$

Solution

(a) $\int_{-1}^{1}(2x - 3)\, dx = [x^2 - 3x]_{-1}^{1}$

$= ((1)^2 - 3(1)) - ((-1)^2 - 3(-1))$

$=(-2) - (4)$

$=-6$

(b) $\int_{1/4}^{1/2}\left(\frac{1}{x^3}\right) dx = \int_{1/4}^{1/2}(x^{-3})\, dx$

$=\left[-\frac{1}{2x^2}\right]_{1/4}^{1/2}$

$=\left(-\frac{1}{2\left(\frac{1}{2}\right)^2}\right) - \left(-\frac{1}{2\left(\frac{1}{4}\right)^2}\right)$

$= -2 + 8$

$=6$

(c) $\int_{-1}^{1}(3x^2 - 5)\,dx = [x^3 - 5x]_{-1}^{1}$

$$= (1 - 5) — 1 + 5$$
$$= -4 - 4$$
$$= -8$$

(d) $\int_2^8 \left(x - \frac{3}{\sqrt{x}}\right) dx = \int_2^8 \left(x - 3x^{-\frac{1}{2}}\right) dx$

$$= \left[\frac{x^2}{2} - 6x^{\frac{1}{2}}\right]_2^8$$
$$= \left(32 - 6\sqrt{2}\right) - \left(2 - 6\sqrt{2}\right)$$
$$= \left(32 - 12\sqrt{2}\right) - \left(2 - 6\sqrt{2}\right)$$
$$= 30 - 6\sqrt{2}$$
$$= 21.5147$$

9.3 Application of Integration

These are some of the situations where integration is called into play

Gradient functions

Here we are given a gradient function and required to find the original function from which it was obtained.

Example 5

Given that $\frac{dy}{dx} = 4$. Find y in terms of x, given that $y = 10$ when $x = -2$. What does this represent geometrically?

Solution

$\Rightarrow \frac{dy}{dx} = 4;\ \therefore y = 4x + c$

$when\ x = -2\ and\ y = 10$

$\Rightarrow 10 = 4(-2) + c;\ \ c = 18$

$\therefore y = 4x + 18$ and this represents a straight line with gradient 4 and y-intercept 18.

Example 6

The gradient of a curve at any point (x, y) on the curve is given by $(7x - 1)$. If the curve passes through the point $(1, 4)$, find the equation of the curve.

Solution

$\Rightarrow \frac{dy}{dx} = (7x - 1);\ \therefore y = \frac{7x^2}{2}x - x + c$

$At\ (1, 4); x = 1\ and\ y = 4$

$\Rightarrow 4 = \frac{7(1)^2}{2} - (1) + c; c = 3/2$

$\therefore y = \frac{7x^2}{2}x - x + \frac{3}{2} \;\; or\; 2y = 7x^2 - 2x + 3$

Example 7

At any point (x, y) on a certain curve, $\frac{dy}{dx} = (3x-2)(x+2)$. Given that it passes through $(-1,\ 1)$ find the equation of the curve. Find and distinguish between the turning points of the curve.

Solution

$\Rightarrow \frac{dy}{dx} = (3x-2)(x+2)$

$\Rightarrow \frac{dy}{dx} = 3x^2 + 4x - 4; \;\; \therefore y = x^3 + 2x^2 - 4x + A$

$when\ x = -1\ and\ y = 1$

$\Rightarrow 1 = (-1)^3 + 2(-1)^2 - 4(-1) + A; \;\; A = -4$

$\therefore y = x^3 + 2x^2 - 4x - 4$

At the turning point $\frac{dy}{dx} = 0$

$\Rightarrow (3x-2)(x+2) = 0; x = 2/3\ \ and\ x = -2$

When $x = 2/3\,; y = \left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) - 4 = -\frac{100}{27}; \;\; \left(\frac{2}{3}, -\frac{100}{27}\right)$

When $x = 2; y = (-2)^3 + 2(-2)^2 - 4(-2) - 4 = 4; \;\; (-2, 4)$

Using the 2nd derivative test:

$\frac{d^2y}{dx^2} = 6x + 4$

$when\ x = 2/3\,; \frac{d^2y}{dx^2} = 6\left(\frac{2}{3}\right) + 4 = 8 > 0\ Hence\ \left(\frac{2}{3}, -\frac{100}{27}\right)$ is a minimum

$when\ x = -2; \frac{d^2y}{dx^2} = 6(-2) + 4 = -8 > 0\ Hence\ (-2, 4)$ is a maximum

Example 8

Given that $f''(x) = 2 - \frac{2}{\sqrt{x^3}}$ and $f'(1) = 0$, find $f'(x)$. Given that $f(1) = 8$. Find $f(x)$

Solution

$\Rightarrow 2 - 2x^{-\frac{3}{2}}$

$\Rightarrow f'(x) = \int f''(x)\, dx$

$\Rightarrow f'(x) = \int \left(2 - \frac{2}{\sqrt{x^3}}\right) dx$

$\Rightarrow f'(x) = 2x - \frac{2x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$

$\qquad\qquad = 2x + \frac{4}{\sqrt{x}} + c$

$But\ f'(1) = 0$

$\Rightarrow 0 = 2(1) + \frac{4}{\sqrt{1}} + c; \;\; c = -6$

$\therefore \boldsymbol{f'(x) = 2x + \frac{4}{\sqrt{x}} - 6}$

$\Rightarrow f'(x) = 2x + 4x^{-\frac{1}{2}} - 6$

$\Rightarrow f(x) = \int f'(x)\,dx$

$= \int \left(2x + 4x^{-\frac{1}{2}} - 6 \quad\right) dx$

$= x^2 + \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} - 6x + A$

$f(x) = x^2 + 8\sqrt{x} - 6x + A$

$when\ f(1) = 8$

$\Rightarrow 8 = 1^2 + 8\sqrt{1} - 6(1) + A;\ \ A = 5$

$\therefore \boldsymbol{f(x) = x^2 + 8\sqrt{x} - 6x + 5}$

Displacement, Velocity and Acceleration

The quantities displacement (r) or (s), the velocity (v) and acceleration (a) with respect to time (t) can be related using differentiation.

$\Rightarrow \frac{ds}{dt} = v; then\ s = \int v\,dt$

$\Rightarrow \frac{dv}{dt} = a; then\ v = \int a\,dt$

When **s**, **v** and **a** are given vectors then we should observe the vector symbols.

Example 9

A stone is thrown vertically downwards at 20m/s from the top of a cliff. Assuming that gravity produces on it an acceleration of 9.8 ms^{-2}, deduce from the differential equation $\frac{dy}{dt} = 9.8$, expressions for its velocity and position vector t s later.

Solution

$\Rightarrow dv = 9.8dt$

$\Rightarrow \int dv = \int 9.8\,dt$

$\Rightarrow v = 9.8t + c$; but $when\ t = 0, v = 20$

$\Rightarrow 20 = 9.8(0) + c;\ \ c = 20$

$\therefore \boldsymbol{v = 9.8t + 20}$

$\Rightarrow \frac{ds}{dt} = 9.8t + 20$

$\Rightarrow ds = (9.8t + 20)dt$

$\Rightarrow \int ds = \int (9.8t + 20)\, dt$

$\Rightarrow s = 4.9t^2 + 20t + A\,;\ but\ when\ t = 0, s = 0$

$\Rightarrow 0 = 4.9(0)^2 + 20(0) + A; \quad A = 0$

$\therefore \boldsymbol{s = 4.9t^2 + 20t}$

Exercise 9.1

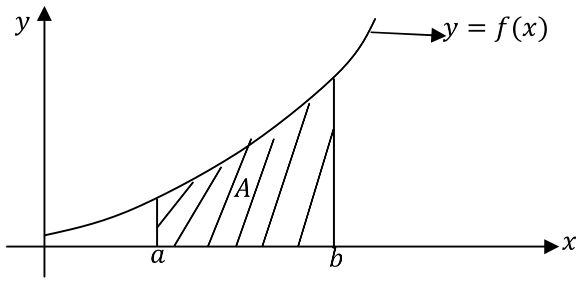
1. Find an expression for y if $\frac{dy}{dx}$ is given by;
   (a) $3x^2$ (b) $2x$ (c) x^3 (d) 5 (e) $\sqrt{x}$ (f) $-\frac{4}{x^3}$ (g) $2x^3 + 3x^2$ (h) $5x + 1$ $(i)x(4 - 3x)$
2. Integrate the following with respect to x;
   (a) $2x^2(3 - 4x)$ (b) $\frac{3x^4+6}{x^2}$ (c) $7 - 2x$ (d) $4x^3 + 3x^2 + 2x + 1$
   (e) $x^4 + x^2 + \frac{1}{x^2} + \frac{1}{x^4}$
3. Find;
   (a) $\int 12\, dx$ (b) $\int (x^3 + x)\, dx$ (c) $\int x(x + 1)\, dx$ (d) $\int (x + 6)(x - 4)\, dx$ (e) $\int \left(10x^4 + 8x^3 - \frac{6}{x^2}\right) dx$ (f) $\frac{(1-3t)}{\sqrt{t}}\, dt$ (g) $\int \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right) dx$
4. Evaluate (a) $[3t + 8]_2^5$ (b) $[3t^2 - t + k]_1^4$
5. Evaluate the following;
   (a) $\int_{-2}^{2} (x^2 + 4x)\, dx$ (b) $\int_0^1 (x^3 - 4x^2 + 3x)\, dx$ (c) $\int_1^4 \left(3 - \frac{1}{\sqrt{x}}\right) dx$ (d) $\int_1^{16} \left(\frac{\sqrt{x}-4}{\sqrt{x}}\right) dx$
   (e) $\int_{-2}^{4} (x - 2)^2\, dx$ $Ans\left((a)\frac{16}{3}\,(b)\frac{5}{12}\,(c) - 9\ (e) - 4.5\right)$
6. The gradient of a curve at the point (x, y) on the curve is given by $6x$. If the curve passes through the point $(1, 4)$, find the equation of the curve. $Ans(y = 3x^2 + 1)$
7. Find the equation of the curve passing through the point $(-2, 6)$ and having the gradient function $3x^2 - 2$. $Ans(y = x^3 - 2x + 10)$
8. The gradient of a curve at the point (x, y) is given by $(3x^2 + 8)$. If the curve and the line $2x - y - 1 = 0$ cut the y-axis at the same point, find the equation of the curve. $Ans(y = x^3 + 8x - 1)$
9. The gradient function of a curve is given by $(2x - 3)$ and the curve cuts the x-axis at two points $A(5, 0)$ and B. Find the equation of the curve and the coordinates of the curve. $Ans(y = x^2 - 3x - 10; B(-2,0))$
10. The gradient of a curve at the point (x, y) on the curve is given by $(2x - 4)$. If the minimum value of y is 3, find the equation of the curve.
11. The gradient of a curve at the point (x, y) is $12x^3 - \frac{1}{x^2}$ and the curve passes through the point $(1, 2)$. Find the equation of the curve. $Ans\left(y = 3x^3 + \frac{1}{x} - 2\right)$.

12. The curve with equation $y = ax^2 + bx + c$ passes through the point $P(2, 6)$ and $Q(3, 16)$, and has a gradient of 7 at the point P. Find the values of the constants a, b and c. $Ans(a = 3, b = -5, c = 4)$
13. Find y in terms of x if $\frac{dy}{dx} = 3x^2 - 6x + 2$ and $y = 7$ when $x = 0$.
14. Find v in terms of t given that $\frac{dv}{dt} = 5 - 2kt$ where k is a constant. If $v = 0$ when $t = 0$; when $t = 1$ find the value of k.
15. Find S in terms of t if $\frac{ds}{dt} = 3t - \frac{8}{t^2}$, given that $s = 1\frac{1}{2}, when\ t = 1$
16. Find A in terms of x if $\frac{dA}{dt} = \frac{(3x+1)(x^2-1)}{x^5}$. What is the value of A when $x = 2$, If $A = 0$ when $x = 1$.
17. A function $f(x)$ is such that $f'(x) = 3\sqrt{x} - 5$. Given that $f(4) = 3$, find an expression for $f(x)$.
18. Given that $\frac{d^2y}{dx^2} = 6x + \frac{4}{x^3}$ and that $y = 1$ when $x = 1$ and that $y = 5$ when $x = 2$. Find an expression for y in terms of x.
19. A particle moves in a straight line with velocity $2t^2\ ms^{-1}$, t s after the start. Find the distance moved in the third second. $Ans\left(12\frac{2}{3}m\right)$
20. Find the displacement (s) in terms of time (t) from the following data
    (a) $\frac{ds}{dt} = 3,\ s = 3\ when\ t = 0,\quad Ans(s = 3t + 3)$
    (b) $v = 4t - 1,\ s = 0\ when\ t = 2.\ \ Ans(2t^2 - t - 6)$
    (c) $v = t^2 + 5 - \frac{2}{t^2},\ s = 1/3\ when\ t = 1\quad Ans\left(\frac{1}{3}t^3 + 5t + 2t^{-1} - 7\right)$
21. A body moves in a straight line. At time t seconds its acceleration is given by $a = 6t + 1$. When $t = 0$, the velocity v of the body is 2m/s and its displacement s from the origin O is 1 metre. Find the expressions for v and s in terms of t.
22. If $a = 1 - t$ and when $t = 2, v = 1\ and\ s = 4\frac{2}{3}$. Find the expression for v and s in terms of t.
23. A train runs non-stop between two stations P and Q, and its velocity t hours after leaving P is $(60t - 30t^2)km/h$. Find;
    (a) the distance between P and Q
    (b) the average velocity for the journey
    (c) the maximum velocity attained
    $Ans\big((a)40km\ (b)20km/h\ (c)30km/h\big)$

9.4 Area under a Curve

Area defined between the curve and the x-axis

Suppose A is the area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ as seen from the figure below.



For the interval $x = a\ and\ x = b$

$\Rightarrow A = \int_a^b y\,dx$ is the area under the curve.

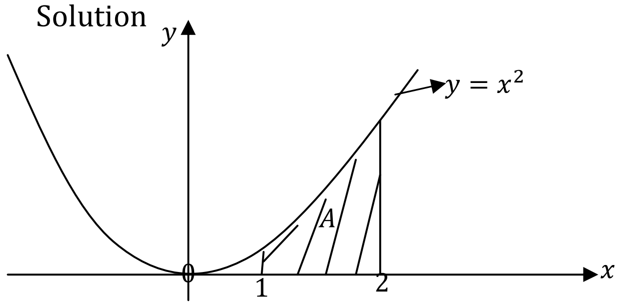
Calculations of the area under the curve

- When calculating the area under the curve the first important step is to make the sketch of the curve.
- We must remember that an area lying above the x-axis will have a positive value, whereas areas lying below the x-axis will be negative.
- In some cases the required area may lie both above and below the x-axis and particular care is needed in these situations.

Example 10

Find the area under the curve $y = x^2$ between $x = 1\ and\ x = 3$

Solution



Let A be the required area, then

$A = \int_1^2 x^2\,dx$

$= \left[\frac{x^3}{3}\right]_1^2$

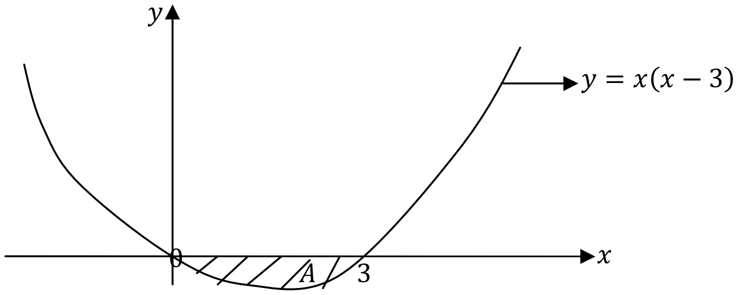
$= \left(9 - \frac{1}{3}\right)$

$= \frac{26}{3} = 8\frac{2}{3}$

Example 11

Find the area below the curve $y = x(x-3)$ and the x-axis

Solution



$\Rightarrow A = \int_0^3 x(x-3)\,dx$

$= \int_0^3 (x^2 - 3x)\,dx$

$= \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3$

$= \left(9 - \frac{27}{2}\right) - (0)$

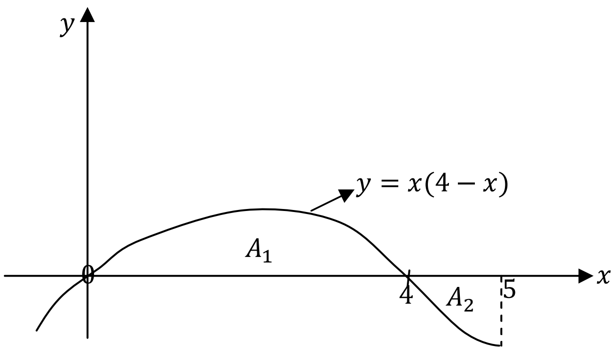
$= -4\frac{1}{2}$

The area has a negative sign as was anticipated because the area in below the x-axis but the numerical value is $\mathbf{4\frac{1}{2}}$ **sq. units.**

Example 12

Find the area between the curve $y = x(4-x)$ and the x-axis for $x = 0$ to $x = 5$

Solution



Area one

$\Rightarrow A_1 = \int_0^4 x(4-x)\,dx$

$= \int_0^4 (4x - x^2)\,dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$

$= \left(32 - \frac{64}{3}\right) - (0)$

$= \frac{32}{3} = 10\frac{2}{3}$

Area two

$\Longrightarrow A_2 = \int_4^5 x(4-x)\,dx$

$= \int_4^5 (4x - x^2)\,dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_4^5$

$= \left(50 - \frac{125}{3}\right) - \left(32 - \frac{64}{3}\right)$

$= -2\frac{1}{3}$

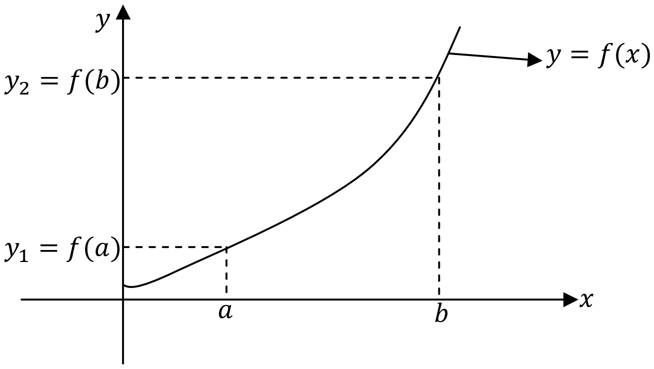
The total area under the curve between $x = 0$ and $x = 5$ is given by the sum of the numerical values of the two areas

$\therefore$ The required area = $\mathbf{10\frac{2}{3} + 2\frac{1}{3} = 13}$ ***sq. units***

NB: If we use $\int_0^5 x(4-x)\,dx$ we obtain $8\frac{1}{3}$ which is not similar to the previous.

Area defined by a curve and the y-axis

Suppose we want to find the area between the curve $y = f(x)$ and the y-axis



$\Longrightarrow A = \int_{f(a)}^{f(b)} x\,dy$

In this case, x must be expressed as a function of y before we can integrate.

Example 13

Find the area between the curve $y = x^2$ and the y-axis between $y = 1$ and $y = 4$

Solution

$\Longrightarrow A = \int_1^4 x\,dy$

$= \int_1^4 y^{\frac{1}{2}}\,dy$

$= \left[\frac{2y^{\frac{3}{2}}}{3}\right]_1^4$

$= \frac{2(4)^{\frac{3}{2}}}{3} - \frac{2}{3}$

$= \frac{16}{3} - \frac{2}{3}$

$= \frac{14}{3}$

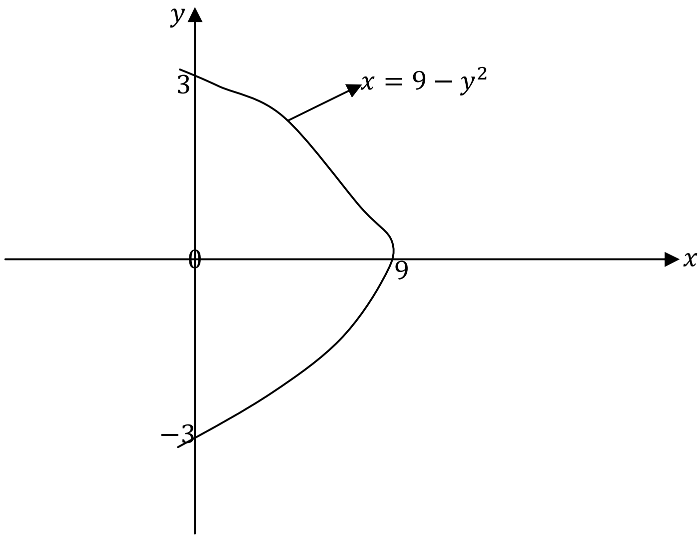
Example 14

Find the area enclosed between the curve $y^2 = 9 - x$ and the y-axis

Solution

When $y = 0, x = 9$; $(9,0)$

When $x = 0, y = \pm 3$; $(0,-3)$ *and* $(0,3)$



$\Rightarrow A = \int_{-3}^{3}(9 - y^2)\,dy = 2\int_{0}^{3}(9 - y^2)\,dy$ Note: $\int_{-a}^{a} x\,dy = 2\int_{0}^{a} x\,dy$

$= 2\left[9y - \frac{y^3}{3}\right]_0^3$

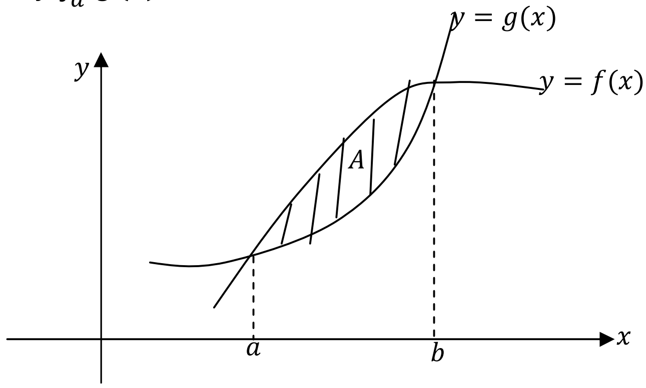
$= 2\{(27 - 9) - (0)\}$

$= 36$

Area defined by two curves:

- An area can be defined by two curves and in this case it is essential to make a sketch and to determine the point of intersection of the two curves.
- Suppose the curves $y = f(x)$ and $y = g(x)$ intersect at the point where $x = a$ and $x = b$
- The area between the curve $y = f(x)$ and the $x - axis$ from $x = a$ to $x = b$ is given by $\int_a^b f(x)\,dx$.

- The area between the curve $y = g(x)$ and the $x - axis$ from $x = a$ to $x = b$ is given by $\int_a^b g(x)\,dx$



The shaded area between the two curves is given by;

$A = \int_a^b f(x)dx - \int_a^b g(x)dx$

$\quad = \int_a^b [f(x) - g(x)]\,dx$

- This can be obtained by saying the upper curve $f(x)$ minus the lower curve $g(x)$.
- The two functions can be either both curves or a curve and a line.

Example 15

Find the area enclosed between the curves $y = x^2 + 2x + 2$ and $y = -x^2 + 2x + 10$

Solution

We must first find the points of intersection of the two curves.

The x-coordinates of the points of intersection satisfy the equation

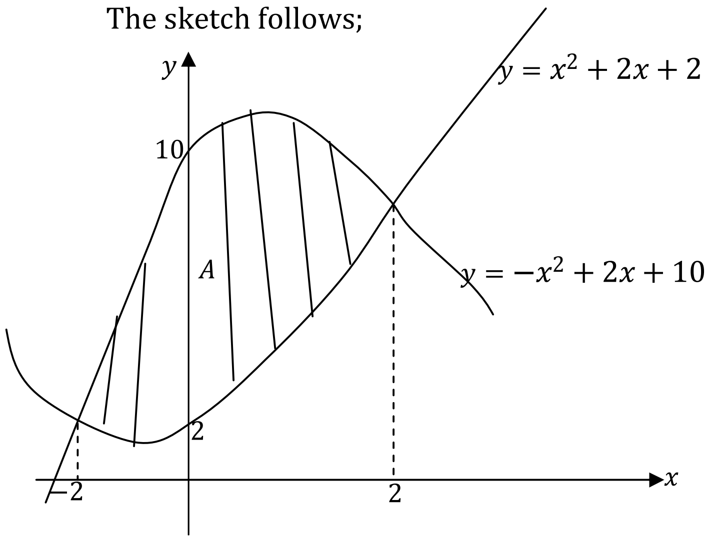
$x^2 + 2x + 2 = -x^2 + 2x + 10$

$\Rightarrow 2x^2 - 8 = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

The sketch follows;



$\Rightarrow A = \int_{-2}^{2} \{(-x^2 + 2x + 10) - (x^2 + 2x + 2)\}\,dx$

$\quad = \int_{-2}^{2} (-2x^2 + 8)\,dx$

$= \left[-\frac{-2x^3}{3} + 8x\right]_{-2}^{2}$
$= \left(-\frac{16}{3} + 16\right) - \left(\frac{16}{3} - 16\right)$
$= \frac{32}{3} + \frac{32}{3}$
$= \frac{64}{3}$

Example 16

Find the area enclosed between the curve $y = x^2 - 2x - 3$ and the line $y = x + 1$

Solution

We must first find the points of intersection of the two curves.

The x-coordinates of the points of intersection satisfy the equation

$x^2 - 2x - 3 = x + 1$
$\Rightarrow x^2 - 3x - 4 = 0$
$\Rightarrow (x - 4)(x + 1) = 0$
$\Rightarrow x = 4, x = -1$

The sketch follows;

For the curve $y = x^2 - 2x - 3$

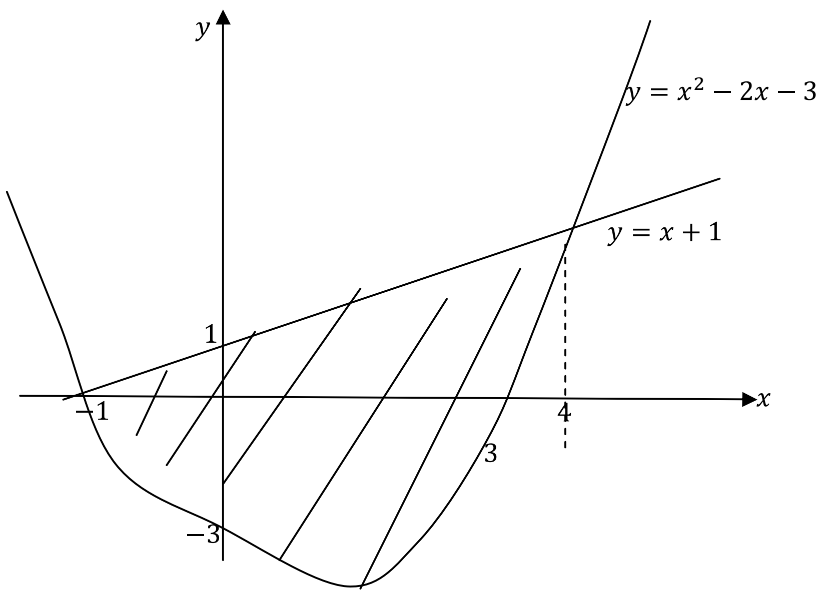
When $x = 0, y = -3; (0, -3)$

When $y = 0;\ x^2 - 2x - 3 = 0 \Leftrightarrow (x - 3)(x + 1) = 0; (3, 0), (-1, 0)$

For the line $y = x + 1$

When $x = 0; y = 1; (0, 1)$

When $y = 0, x = -1;\ (-1, 0)$



$A = \int_{-1}^{4}\{(x + 1) - (x^2 - 2x - 3)\}\, dx$
$= \int_{-1}^{4}(-x^2 + 3x + 4)\, dx$

$= \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 4x\right]_{-1}^{4}$
$= \left(-\frac{64}{3} + 24 + 16\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right)$
$= \frac{56}{3} - \left(-\frac{13}{6}\right)$
$= \frac{125}{6}$

Exercise 9.2

1. Find the area enclosed between the curve $y = x^2 + 4x$ and the x-axis from
   (a) $x = -2\ to\ x = 0$ (b) $x = -2\ to\ x = 2$
   $Ans\left((a)\frac{16}{3}\ (b)\frac{32}{3}\right)$
2. Sketch the curve of $y = x^3 - 4x^2 + 3x$ and find the area enclosed between the curve and the x-axis from $x = 0\ to\ x = 3$. $Ans\left(\frac{37}{12}\right)$
3. Find the area enclosed by $y = 4x - x^2, x = 1, x = 2$ and the x-axis. $Ans\left(3\frac{2}{3}\right)$
4. Find the area enclosed by the curve $y^2 = 4x$ an the straight lines $x = 1\ and\ x = 4$
5. Find the area enclosed by the x-axis, the curve $y = 3x^2 + 2$ and the straight lines $x = 3$ and $x = 5$.
6. Find the area under the curve $y = x^2(x - 2)$
   (a) From $x = 0\ to\ x = 2$
   (b) From $x = 2\ to\ x = 8/3$
   $Ans\left((a) - 1\frac{1}{3}\ (b)1\frac{1}{3}\right)$
7. Sketch the curve $y = x^2 - 5x + 6$ and find the area cut off below the x-axis.
8. Sketch the curve $y = x(x + 1)(2 - x)$ and find the area of each of the segments cut off by the x-axis.
9. Find the total area enclosed between $y = (x^2 - 1)(x - 3)$
10. Find the area between the curve $y = 2x^2$ and the y-axis cut off by the lines parallel to the x-axis through the points on the curve where $x = 1\ and\ x = 3$
11. Find the area enclosed between the curve $y^2 = 4 - x$ and the y-axis
12. Find the area enclosed between the curves $y = 2x^2 + 3$ and $y = 10x - x^2$. $Ans\left(9\frac{13}{27}\right)$
13. The line $y = 3x + 1$ meets the curve $y = x^2 + 3$ at the points P and Q.
    (a) Find the coordinates of P and Q
    (b) Sketch the line and the curve on the same pair of axes.
    (c) Calculate the area of the finite region bounded by the line and the curve.
14. Find the area enclosed between the two curves $y = 4 - x^2$ and $y = x^2 - 2x$. $Ans(9)$
15. The curve $y = x^2 - 1$ cuts the x-axis at the points P and Q and meets the line $y = x + 1$ at the point R.
    (a) Calculate the coordinates of P, Q and R
    (b) Find the area of the region enclosed between the curve and the x-axis.
16. The curve $y = x^2 + 16$ meets the curve $y = x(12 - x)$ at the points C and D.

(a) Find the coordinates of C and D.
(b) Sketch the two curves on the same pair of axes.
(c) Calculate the area bounded by the two curves.

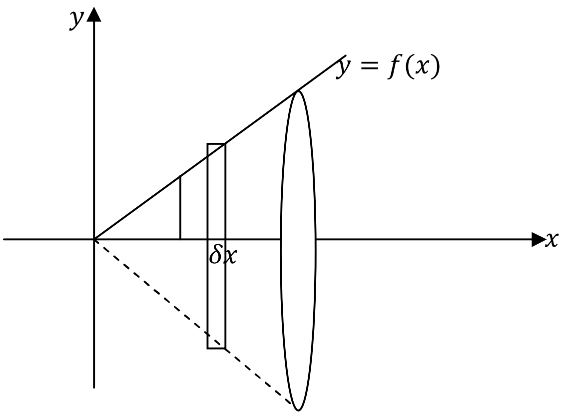
17. (a) Sketch, on the same diagram, the curves $y = x^2 - 5x$ and $y = 3 - x^2$, and find their points of intersection.
(b) Find the area of the region bounded by the two curves.
$Ans\left(14\frac{7}{24}\right)$

9.5 Volume of Revolution

- This involves finding volumes of solids generated.

Rotation about the x-axis

- Suppose that the area bounded by the curve $y = f(x)$, the $x - axis$ and the ordinates $x = a$ and $x = b$ is rotated about the x-axis through one revolution.
- The volume of the solid formed is given by $\boldsymbol{V = \pi \int_a^b y^2\, dx}$, this definite integral can be obtained and V can be evaluated.



Rotation can be;

(a) Through one revolution or four right angles; $\boldsymbol{V = \pi \int_a^b y^2\, dx}$

(b) Through half a revolution or two right angles; $\boldsymbol{V = \frac{\pi}{2} \int_a^b y^2\, dx}$

(c) Through quarter a revolution or one right angle; $\boldsymbol{V = \frac{\pi}{4} \int_a^b y^2\, dx}$

Example 17

Find the volume of the solid generated by rotating about the x-axis the area under $y = \frac{3}{4}x$ from $x = 0\ to\ x = 4$

Solution

$V = \pi \int_0^4 \left(\frac{3}{4}x\right)^2 dx$

$= \pi \int_0^4 \frac{9}{16}x^2\, dx$

$= \pi \left[\frac{9x^3}{48}\right]_0^4$

$= \pi\left(\frac{9\times4^3}{48}\right)$
$= 12\pi$

Qn. Find the volume of the solid generated by rotating about the x-axis

(a) the area under $y = x^2$ from $x = 1\ to\ x = 2$

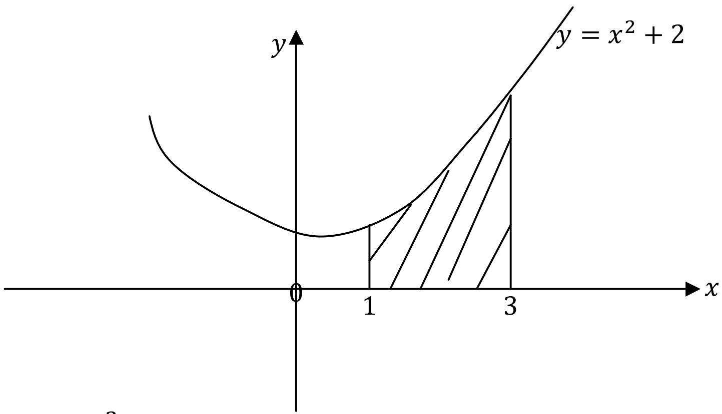
(b) the area under $y = x^2 + 1$ from $x = -1\ to\ x = 1$

$Ans\left((a)\frac{31\pi}{5}(b)\frac{56\pi}{15}\right)$

Example 18

Find the volume of the solid formed when the area between the curve $y = x^2 + 2$ and the x-axis from $x = 1$ to $x = 3$ is rotated through 2π radians about the x-axis.

Solution



$V = \pi\int_1^3 y^2\,dx$

Now $y^2 = (x^2 + 2)^2 = x^4 + 4x^2 + 4$ therefore

$V = \pi\int_1^3 (x^4 + 4x^2 + 4)\,dx$

$= \pi\left[\frac{x^5}{5} + \frac{4x^3}{3} + 4x\right]_1^3$

$= \left(\frac{(3)^5}{5} + \frac{4(3)^3}{3} + 4(3)\right) - \left(\frac{(1)^5}{5} + \frac{4(1)^3}{3} + 4(1)\right)$

$= \pi\left(\frac{483}{5} - \frac{83}{15}\right)$

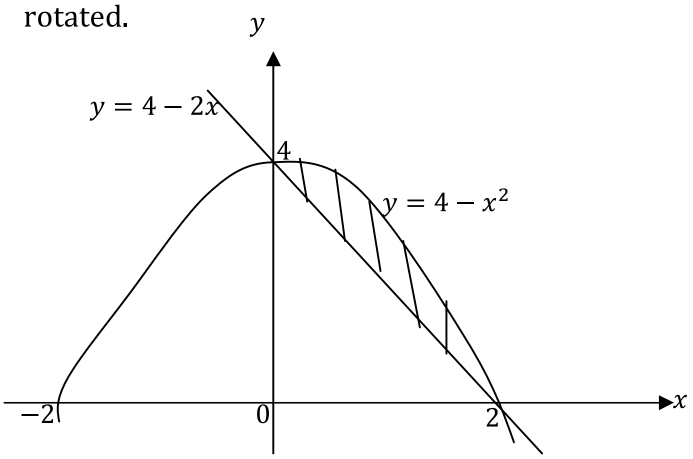
$= \frac{1366\pi}{15}$

Example 19

The area enclosed between the curve $y = 4 - x^2$ and the line $y = 4 - 2x$ is rotated through 2π radians about the x-axis. Find the volume of the solid generated.

Solution

The sketch of both the curve and the line on the same set of axes shows the area to be rotated.



$V = \pi \int_0^2 [(4-x^2)^2 - (4-2x)^2]\, dx$

$= \pi \int_0^2 (x^4 - 12x^2 + 16x)\, dx$

$= \pi \left[\frac{x^5}{5} - 4x^3 + 8x^2\right]_0^2$

$= \left(\frac{(2)^5}{5} - 4(2)^3 + 8(2)^2\right) - (0)$

$= \frac{32\pi}{5}$

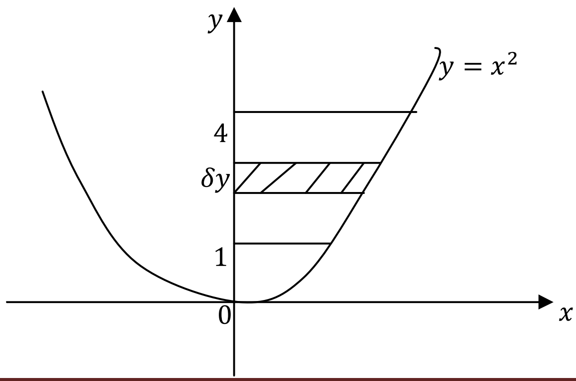
Rotation about the y-axis

- The volume of the solid of revolution formed by rotating an area through one revolution about the y-axis through 2π radians can be found in the similar way to that about the $x - axis$.
- The volume of the solid generated can be given by $V = \pi \int_{y=a}^{y=b} x^2\, dx$ where $x = f(y)$
- The dynamics of rotation apply here also.

Example 20

Find the volume of the solid generated by rotating about the y-axis the area in the first quadrant enclosed by $y = x^2, y = 1, y = 4$ and the y-axis

Solution



$V = \pi \int_1^4 x^2 \, dy$

$= \pi \int_1^4 y \, dy$

$= \pi \left[\frac{y^2}{2}\right]_1^4$

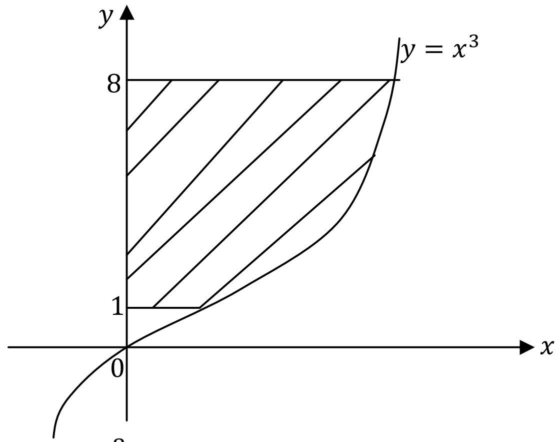
$= \frac{\pi}{2}\left\{\left(\frac{(4)^2}{2}\right) - \left(\frac{(1)^2}{2}\right)\right\}$

$= \frac{15\pi}{2}$

Example 21

Find the volume of the solid formed when the area between the curve $y = x^3$ and the y-axis from $y = 1$ to $y = 8$ is rotated through 2π radians about the y-axis.

Solution



$V = \pi \int_1^8 x^2 \, dy$

Now $y = x^3$, therefore

$y^{\frac{2}{3}} = (x^3)^{\frac{2}{3}}$

$\therefore y^{\frac{2}{3}} = x^2$

$= \pi \int_1^8 y^{\frac{2}{3}} \, dy$

$= \pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_1^8$

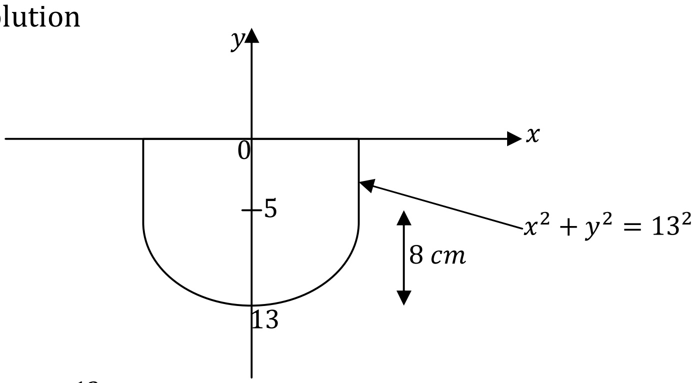
$= \pi \left\{\left(\frac{3}{5}(8)^{\frac{5}{3}}\right) - \left(\frac{3}{5}(1)^{\frac{5}{3}}\right)\right\}$

$= \frac{93\pi}{5}$

Example 22

A hemispherical bowl of internal radius 13 cm contains water to a maximum depth of 8cm. find the volume of water.

Solution



$V = \pi \int_5^{13} x^2\, dy$

$= \pi \int_5^{13} (13^2 - y^2)\, dy$

$= \left[169y - \frac{y^3}{3}\right]_5^{13}$

$= \left(169(13) - \frac{(13)^3}{3}\right) - \left(169(5) - \frac{(5)^3}{3}\right)$

$= 661\frac{1}{3}\pi\ cm^3$

Rotation about any given line

For this we consider the point of intersection between the line and the curve to obtain the limits.

Example 23

The area of the segment cut off by $y = 5$ from the curve $y = x^2 + 1$ is rotated about $y = 5$; find the volume generated.

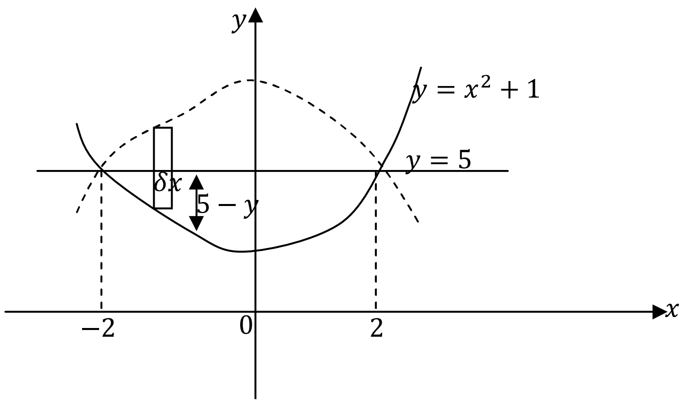
Solution

The point of intersection occur when

$x^2 + 1 = 5$

$\Longrightarrow x^2 = 4$

$\therefore x = 2\ or - 2$



$V = \pi \int_{-2}^{2} \left(5 - (x^2 + 1)\right)^2 dx$

$= \pi \int_{-2}^{2} (4 - x^2)^2 dx$

$= \pi \int_{-2}^{2} (16 - 8x^2 + x^4) dx$

$= \pi \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2}$

$= \pi \left\{\left(16(2) - \frac{8(2)^3}{3} + \frac{(2)^5}{5}\right) - \left(16(-2) - \frac{8(-2)^3}{3} + \frac{(-2)^5}{5}\right)\right\}$

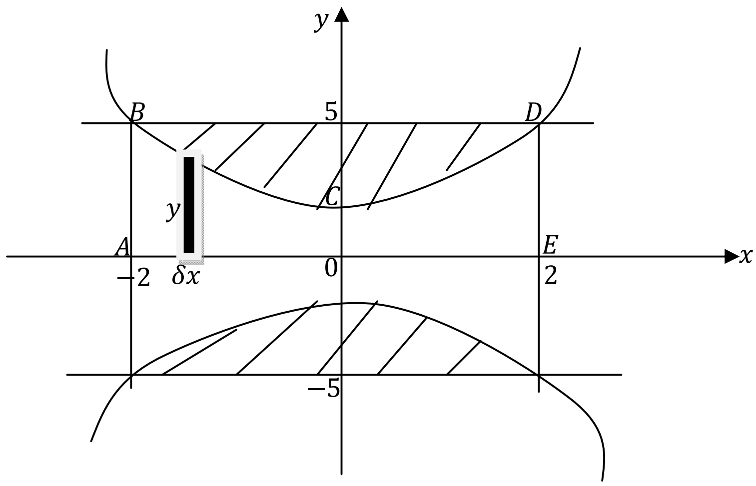
$= 34\frac{2}{15}\pi$

Example 24

The area of the segment cut off by $y = 5$ from the curve $y = x^2 + 1$ is rotated about the x-axis; find the volume generated.

Solution

The solid generated is a cylinder fully open at each end, but with the internal diameter decreasing towards the middle; its volume is found by subtracting the volume of the cavity from the volume of the solid cylinder of the same external dimensions.



V_1 = the volume generated by rotation, about the x-axis, of the rectangle ABCDE

$= \pi r^2 h = \pi(5^2) \times 4 = 100\pi$

V_2 = the volume generated by rotation, about the x-axis, of the area under $y = x^2 + 1$ from $x = -2\ to\ x = 2$

$= \pi \int_{-2}^{2} y^2 dx$

$= \pi \int_{-2}^{2} (x^2 + 1)^2 dx$

$= \pi \int_{-2}^{2} (x^4 + 2x^2 + 1) dx$

$= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_{-2}^{2}$

$= \pi \left\{\left(\frac{(2)^5}{5} + \frac{2(2)^3}{3} + (2)\right) - \left(\frac{(-2)^5}{5} + \frac{2(-2)^3}{3} + (-2)\right)\right\}$

$= 27\frac{7}{15}\pi$

Required volume $V = V_1 - V_2$

$$= 100\pi - 27\frac{7}{15}\pi$$

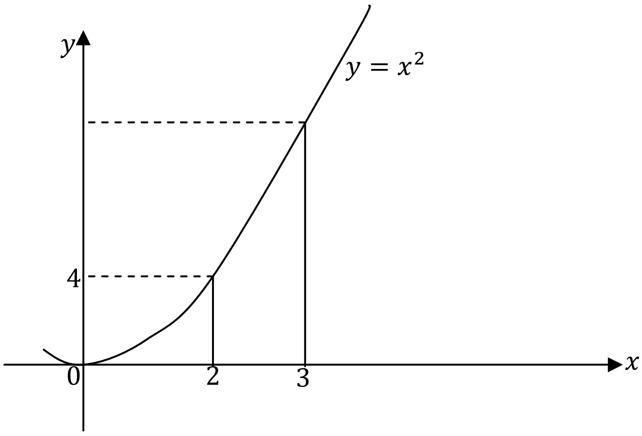
$$= 72\frac{8}{15}\pi$$

Example 25

Find the volume generated when the region bounded by the curve $y = x^2$, the x-axis and the line $x = 2$ is rotated through 180^0 about the line $x = 3$.

Solution

The required volume, V, is found by first calculating the volume, V_1, of the solid generated when the area between the curve $y = x^2$, the x-axis and the line $x = 3$ is rotated through 180^0 about the line $x = 3$ and then subtracting the volume, V_2, of the half-cylinder with radius $(3 - 2) = 1$ and height 4.



$$V_1 = \frac{\pi}{2}\int_0^4 (3 - x)^2\, dy$$

$$= \frac{\pi}{2}\int_0^4 \left(3 - \sqrt{y}\right)^2 dy$$

$$= \frac{\pi}{2}\int_0^4 \left(9 - 6\sqrt{y} + y\right) dy$$

$$= \frac{\pi}{2}\left[9y - 4y^{\frac{3}{2}} + \frac{y^2}{2}\right]_0^4$$

$$= \frac{\pi}{2}\left\{\left(9(4) - 4(4)^{\frac{3}{2}} + \frac{(4)^2}{2}\right) - (0)\right\}$$

$$= 6\pi$$

The volume, V_2, of the half-cylinder is given by

$$V_2 = \frac{\pi}{2}(1^2)4$$

$$= 2\pi$$

Therefore, the required volume, V, is given by

$$V = V_1 - V_2$$

$$= 6\pi - 2\pi$$

$$= 4\pi$$

9.6 Mean Value of a Function

- If we consider the area under the curve $y = f(x)$ from $x = a$ to $x = b$

$\Rightarrow A = \int_a^b y\, dx$

- The mean value of $f(x)$ in the range $a \leq x \leq b$ is given by

  Mean value $= \frac{1}{(b-a)} \int_a^b y\, dx$

Example 26

Find the mean value with respect to x of the function $(5x^2 - 4x)$ for $1 \leq x \leq 3$

Solution

$$Mean\ value = \frac{1}{(3-1)} \int_1^3 (5x^2 - 4x)\, dx$$
$$= \frac{1}{2}\left[\frac{5x^3}{3} - 2x^2\right]_1^3$$
$$= \frac{1}{2}\left\{\left(\frac{5(3)^3}{3} - 2(3)^2\right) - \left(\frac{5(1)^3}{3} - 2(1)^2\right)\right\}$$
$$= 13\frac{2}{3}$$

Exercise 9.3

1. Find the volume of the solid of revolution formed by rotating the area enclose by the curve $y = x + x^2$, the x-axis and the ordinates $x = 2\ and\ x = 3$ through one revolution.
2. The region bounded by the curve $y = x^2 - 2x$ and the x-axis from $x = 0\ to\ x = 2$, is rotated about the x-axis. Calculate the volume of the solid formed.
3. The region bounded by the curve $y = x^2$, the x-axis and the line $x = 2$, is rotated through one revolution about the x-axis. Find the volume of the solid generated.
4. Find the equation of the chord which joins the points A(-2, 3) and B(0, 15) on the curve $y = 15 - 3x^2$.
   (a) Show that the finite area enclosed by the curve and the chord AB is 4 square units
   (b) Find the volume generated when this area is rotated through 360^0 about the x-axis, leaving your answer in terms of π.
5. Solids of revolution are generated by rotating about the y-axis the area bounded by the arc $y = 2x^2$, the line $y = 8$ and the y-axis.
6. The curve $y = x^2 + 4$ meets the axes of y at the point A and B is the point on the curve where $x = 2$. Find the area between the arc AB, the axes, and the line $x = 2$. If the area is revolved about the x-axis, Prove that the volume swept out is approximately 188.
7. The area enclosed by $y = x^2 - 6x + 18$ and $y = 10$ is rotated about $y = 10$. Find the volume generated.
8. Find the volume generated when each of the areas, bounded by the following curves and the x-axis, is rotated through 360^0 about the x-axis between the given lines.
   (a) $y = x; x = 0\ and\ x = 6$
   (b) $y = \sqrt{x}; x = 0\ and\ x = 4$

(c) $y = 3\sqrt{x};\ \ x = 2\ and\ x = 4$
(d) $y = 5 - x;\ \ x = 2\ and\ x = 5$
(e) $y = \sqrt{x^2 + 3x}; x = 2\ and\ x = 6$
(f) $y = \frac{x^2-2}{x^2}; x = 1/4\ \ and\ x = 1/2$
(g) $y = \frac{1}{x^2};\ \ x = 1\ and\ x = 2$

$Ans\left((a)75\pi(b)8\pi(c)54\pi\ (d)9\pi\ (e)\frac{352\pi}{3}(f)\frac{803\pi}{12}(g)\frac{7\pi}{24}\right)$

9. Find the volume generated when each of the areas in the positive quadrant, bounded by the following curves and lines, is rotated through 360^0 about the y-axis.
(a) $y = \frac{1}{2}x; y = 0\ and\ y = 6$
(b) $y = 3x; y = 3\ and\ y = 6$
(c) $y = \frac{1}{2}x + 3; y = 4\ and\ y = 6$
(d) $y = \sqrt{x^2 + 1}; y = 1\ and\ y = 3$
(e) $y = x^2 + 2; y = 2\ and\ y = 6$

$Ans\left((a)\frac{81\pi}{2}(b)7\pi(c)\frac{104\pi}{3}(d)\frac{20\pi}{3}(e)8\pi\right)$

10. The curve $y = x^2$ meets the line $y = 4$ at the points P and Q.
(a) Find the coordinates of P and Q
(b) Calculate the volume generated when the region bounded by the curve and the line is rotated through 360^0 about the x-axis.

$Ans\left((a)P(-2,4), Q(2,4)(b)\frac{256\pi}{5}\right)$

11. The curve $y = x^2 + 1$ meets the line $y = 2$ at the points A and B.
(a) Find the coordinates of A and B.

The region bounded by the curve and the line is rotated through 360^0 about the x-axis.

(b) Calculate the volume of the solid generated. $Ans\left((a)A(-1,2), B(1,2)(b)\frac{64\pi}{15}\right)$

12. The region bounded by the line $y = x + 1, y = 3$ and the y-axis is rotated through 360^0 about the x-axis. Calculate the volume of the solid generated. $Ans\left(\frac{28\pi}{3}\right)$

13. Calculate the volume generated when the region bounded by the curve $y = \frac{4}{x}$ and the line $x = 1$ and $y = 1$ is rotated through 360^0 about the x-axis. $Ans(9\pi)$

14. The region R is bounded by the curve $y = x^2 + 2$, the line $x = 1$ and the x-axis and y-axis. Calculate the volume of the solid generated when R is rotated through 360^0 about the y-axis. $Ans\left(\frac{5\pi}{2}\right)$

15. The line $y = 3x$ meets the curve $y = x^2$ at the point O and P.
(a) Calculate the coordinates of P.

(b) Find the volume of the solid generated when the area enclosed by the line and the curve is rotated through 360^0 about (i) x-axis (ii) y-axis

$Ans\left((a)P(3,9)(b)(i)\frac{162\pi}{5}(ii)\frac{27\pi}{2}\right)$

16. (a) On the same set axes sketch the graphs of the curves $y = x(1-x)$ *and* $y = 2x(1-x)$

(b) Calculate the volume generated when the finite region bounded by the two curves is rotated through 360^0 about the x-axis. $Ans\left(\frac{\pi}{10}\right)$

17. The curve $y = (x-2)(x-4)$ meets the line $y = 8$ at the points A and B.

(a) Find the coordinates of A and B.

The region bounded by the curve and the line $y = 8$ is rotated through 360^0 about the line $y = 8$

(b) Calculate the volume of the solid generated. $Ans\left((a)A(0,8), B(6,8)\ (b)\frac{1296\pi}{5}\right)$

18. The area in the positive quadrant, bounded by the curve $y = x^2$, the x-axis and the line $x = 3$, is rotated through 360^0 about the line $x = 3$. Calculate the volume of the solid generated. $Ans\left(\frac{63\pi}{2}\right)$

19. Find the mean value with respect to x of the function $(x^4 - 2x + 7)$ for $0 \le x \le 1$

CHAPTER 10

SERIES

- <u>A series</u> is a set of numbers each of which can be obtained from some definite law.
- A series is also called a progression.
- <u>A Sequence</u> is a list written down in a definite order and there is a simple rule by which the terms are obtained.
- Each of the numbers forming the set is called a **term** of the series.
- E.g. 3,6,9,12,...
  The first term is 3, the second term is 6, the nth term is $3n$
  We write the nth term as u_n
  $\Rightarrow u_n = 3n$ for $n \ge 1$

Example 1

Write down the first three terms of the sequence whose nth term is given by
$u_n = n^2 + 6n$ for $n \ge 1$

Solution

Let $n = 1$, then $u_1 = (1)^2 + 6(1) = 7$
Let $n = 2$, then $u_2 = (2)^2 + 6(2) = 16$
Let $n = 3$, then $u_3 = (3)^2 + 6(3) = 27$
The first three terms of the sequence are $7, 16, 27$.

Example 2

Write down an expression, in terms of n for the nth term of the sequence 5, 9, 13, 17, ...

Solution

Finding the difference between each of the consecutive terms gives 4.

Consider the sequence defined by $u_n = 4n$

4, 8, 12, 16, ...

It is clear that we require the sequence defined by $u_n = 4n + 1$

- A series may end after a finite number of times in which case it is called <u>a finite series;</u> or it may be considered not to end its called <u>an infinite series</u>
- There are two important progressions under consideration and these are:-

(a) Arithmetic progression **(A.P)** and

(b) Geometric progression **(G.P)**

10.1 The Arithmetic Progression

- A series in which each term is obtained from the preceding one by adding (or subtracting) a constant quantity is called an Arithmetical progression (A.P).
- The terms of an A.P are **a**, $(\mathbf{a + d})$, $(\mathbf{a + 2d})$, $(\mathbf{a + 3d})$. Where **a** is the <u>first term</u> and **d** is the <u>common difference.</u>
- The difference between each term and the preceding one is called the <u>common difference (d)</u>
- When three quantities are in A.P the middle one is called the <u>Arithmetic mean</u> of the other two i.e. for **a, b, c** then $\mathbf{b} = \frac{\mathbf{a+c}}{\mathbf{2}}$ which is the arithmetic mean.
- In the series **a**, $(\mathbf{a + d})$, $(\mathbf{a + 2d})$, $(\mathbf{a + 3d})$, the coefficient of **d** in any term is one less than the number of the term in the series. Thus (a+3d) is the fourth term.
- If then the series consists of n terms and **L** denotes the last term or **n**th term $\mathbf{L = a + (n - 1)d}$
- The sum of the first **n** terms of an A.P is given by $\mathbf{S_n = \frac{n}{2}\{2a + (n - 1)d\}}$ **or** $\boldsymbol{S_n = \frac{n}{2}(a + L)}$

Proof

Consider the sum of the first n terms of an A.P:

$S_n = a \quad + \quad (a + d) \quad + \cdots + \quad [a + (n-1)d]$ ....(1)

Writing the term on the right in reverse order gives

$S_n = [a + (n-1)d] + \quad [a + (n-2)d] \quad + \cdots + \quad (a)$ ....(2)

Adding (1) and (2) gives

$2S_n = \{a + [a + (n-1)d]\} + \{(a + d) + [a + (n-2)d]\} + \cdots + \{[a + (n-1)d] + \text{a}\}$

$= [2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d]$

$= n[2a + (n-1)d]$

$\therefore S_n = \frac{n}{2}[2a + (n-1)d] \ldots \ldots \ldots \ldots \ldots \ldots \ldots .$ ■

Alternatively

$S_n = \frac{n}{2}[a + a + (n-1)d]$

$= \frac{n}{2}(1^{st}\ term + nth\ term)$

$\therefore S_n = \frac{n}{2}(a + L)$

Example 3

1. Write down the sum of the first 100 numbers.
   1+2+3+4+ . . . +97+98+99+100
   100+99+98+97+ . . .+4+3+2+1
   =101+101+101
   The numbers in each column have been added together and since there are 100 terms in the top line, the total is
   100X101=10100 But this is twice the sum required hence the required $=\frac{10100}{2}=$ 5050

Example 4

The sum of the first n terms of a series is given by $S_n = \frac{1}{2}(5n^2 + n)$ for $n \geq 1$

Find u_r, an expression for the rth term of the series

Solution

The rth term can be found by finding the difference between the sum of the first r terms and the sum of the first $(r-1)$ terms;

$\Rightarrow u_r = S_r - S_{(r-1)}$

$= \frac{1}{2}(5r^2 + r) - \frac{1}{2}(5(r-1)^2 + (r-1))$

$= \frac{1}{2}[5r^2 + r - 5(r-1)^2 - (r-1)]$

$= \frac{1}{2}[5r^2 + r - 5(r^2 - 2r + 1) - r + 1]$

$= \frac{1}{2}(10r - 4)$

$\therefore u_r = \frac{1}{2}(10r - 4)$

The rth term of the series is given by $u_r = 5r - 2$

Example 5

Find three numbers in an arithmetical progression such that their sum is 27 and their product is 504

Solution

Let the numbers be $(a-d)$, (a) and (a + d)

Using the sum

$\Rightarrow (a-d) + (a) + (a+d) = 27$

$\Rightarrow 3a = 27; a = 9$

Using the product

$\Rightarrow (9-d)(9)(9+d) = 504$

$\Rightarrow 81 - d^2 = 56; d = \pm 5$

Taking $d = 5$

$\Rightarrow \textit{the terms are } (9-5), 9, (9+5)$

They are 4, 9 and 14.

Taking $d = -5$ gives the terms as 14, 9, 4.

Example 6

The first term of an A.P is 25 and the third term is 19. Find the number of terms in the progression if its sum is 82.

Solution

Given that $a = 25$ and 3^{rd} *term is* 19

$\Rightarrow a + 2d = 19$
$\Rightarrow 25 + 2d = 19; d = -3$
Using $S_n = \frac{n}{2}\{2a + (n-1)d\}$
$\Rightarrow \frac{n}{2}\{2(25) + (n-1)(-3)\} = 82$
$\Rightarrow n(50 - 3n + 3) = 164$
$\Rightarrow -3n^2 + 53n - 164 = 0$
$\Rightarrow n = \frac{-53 \pm \sqrt{(53)^2 - 4(-3)(-164)}}{6}$
$\therefore n = 4,\ n = \frac{41}{3}$
Therefore the value of n is 4.

Example 7
In an A.P the thirteen term is 27 and the seventh term is three times the second term. Find the first term, the common difference and the sum of the first ten terms.
Solution
$a + 12d = 27$ ……………………..(1)
$a + 6d = 3(a + d)$
$\Rightarrow d = \frac{2}{3}a$ ………………………..(2)
Putting (2) into (1)
$\Rightarrow a + 12\left(\frac{2}{3}a\right) = 27; a = 3$ and $d = \frac{2}{3}(3) = 2$
The sum of first 10 terms
$S_{10} = \frac{10}{2}\{2(3) + (10-1)(2)\}$
$= 120$

Example 8
Find the sum of the first 20 terms of the A.P with first term 3 and common difference1/2.
Solution
$n = 20, a = 3, d = 1/2$
$S_{20} = \frac{20}{2}\left\{2(3) + (20-1)\left(\frac{1}{2}\right)\right\}$
$= 155$
Example 9
Find the sum of the series: 6+10+14+…+50
Solution
$a = 6, d = 10 - 6 = 4$
Using $L = a + (n-1)d$
$\Rightarrow 50 = 6 + (n-1)(4);\ n = 12$
$\Rightarrow S_{12} = \frac{12}{2}\{2(6) + (12-1)(4)\}$
$= 336$
Example 10
The first four terms of an AP are 5, 11, 17 and 23. Find the 30^{th} term and the sum of the first 30 terms.
Solution
$a = 5, d = 11 - 5 = 6$

The 30th term is given by

$u_{30} = 5 + (30 - 1)6$

$= 5 + (29)6$

$= 179$

The 30th term is 179

The sum of the first 30 terms is given by

$S_{30} = \frac{30}{2}[2(5) + (30 - 1)(6)]$

$= 15[10 + (29)(6)]$

$= 2760$

The sum of the first 30 terms is 2760

Example 11

An AP has a first term of 2 and nth term of 32. Given that the sum of the first n terms is 357, find n and the common difference of the AP.

Solution

Since the nth term is 32, we have

$a + (n - 1)d = 32$

We also know that $a = 2$. Therefore

$2 + (n - 1)d = 32$

$\Rightarrow (n - 1)d = 30$ ………………(1)

Since the sum of the first n terms is 357, we have

$\frac{n}{2}[2a + (n - 1)d] = 357$

$\Rightarrow \frac{n}{2}[2(2) + 30] = 357$

$\Rightarrow 34n = 714$

$\therefore n = 21$

Substituting $n = 21$ into (1) gives

$(21 - 1)d = 30$

$\therefore d = 3/2$

The value of n is 21 and the common difference is $3/2$.

Example 12

The 3rd, 5th and 8th terms of an AP are $3x + 8, x + 24$ and $x^3 + 15$ respectively. Find the value of x and hence the common difference of the AP.

Solution

Since the 3rd, 5th and 8th terms of an AP are $3x + 8, x + 24$ and $x^3 + 15$,

We have

$a + 2d = 3x + 8$ ……………………(1)

$a + 4d = x + 24$ ……………………(2)

$a + 7d = x^3 + 15$ …………………..(3)

Subtracting (1) form (2) gives

$2d = -2x + 16$

$d = -x + 8$ … …………………………(4)

Subtracting (1) form (3) gives

$5d = x^3 - 3x + 7$ ……………………(5)

Substituting (4) into (5) gives

$5(-x + 8) = x^3 - 3x + 7$

$\Rightarrow -5x + 40 = x^3 - 3x + 7$

$\Rightarrow x^3 + 2x - 33 = 0$
$\Rightarrow (x-3)(x^2 + 3x + 11) = 0$
Solving gives $x = 3\ or\ x^2 + 3x + 11 = 0$. Since $x^2 + 3x + 11 = 0$ has no real solutions, we have $x = 3$.
The common difference of the AP is given by $d = -x + 8 = -3 + 8 = 5$.

Roots of cubic equations
$\Rightarrow$Using $ax^3 + bx^2 + cx + d = 0$
$\Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$
$\Rightarrow (x-\alpha)(x-\beta)(x-\gamma) = 0$ where α, β and γ are the roots of a cubic equation
Expanding and compare the coefficients
$\Rightarrow \alpha + \beta + \gamma = -\frac{b}{a}$
$\Rightarrow \beta\gamma + \gamma\alpha + \alpha\beta = \frac{c}{a}$
$\Rightarrow \alpha\beta\gamma = -\frac{d}{a}$
Example 13
Solve the equation $64x^3 - 240x^2 + 284x - 105 = 0$ given that the roots are in an arithmetic progression.

<u>Solution</u>
Let the terms be $(a-d), (a)$and$(a+d)$ be the terms

$\Rightarrow$Using $ax^3 + bx^2 + cx + d = 0$
$\Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$
$\Rightarrow (x-\alpha)(x-\beta)(x-\gamma) = 0$ where α, β and γ are the roots of a cubic equation
$\Rightarrow \alpha + \beta + \gamma = -\frac{b}{a}$
$\Rightarrow \beta\gamma + \gamma\alpha + \alpha\beta = \frac{c}{a}$
$\Rightarrow \alpha\beta\gamma = -\frac{d}{a}$
Sum of roots
$\Rightarrow (a-d) + (a) + (a+d) = -\frac{240}{64}$
$\Rightarrow a = \frac{5}{4}$
Product of roots
$\Rightarrow (a-d)(a)(a+d) = \frac{105}{64}$
$\Rightarrow \frac{5}{4}\left(\left(\frac{5}{4}\right)^2 - d^2\right) = \frac{105}{64}$
$\Rightarrow d = \pm\frac{2}{4}$
Substituting the values of a and d in the terms, we obtain the roots as $\frac{3}{4}, \frac{5}{4}$and$\frac{7}{4}$

Exercise 10.1
1. The sum of the first n terms of a series is $n(n+2)$. Find the first three terms of the series. $Ans(3, 5, 7)$
2. The sum of the first n terms of a series is $\frac{n}{2}(n+8)$. Find the 1st , 2nd and the 10th terms. $Ans(4.5, 5.5, 13.5)$

3. In an AP, the 1st term is 13 and the 15th term is 111. Find the common difference and the sum of the first 20 terms. $Ans(7,\ 1590)$
4. The 5th term of an AP is 7 and the common difference is 4. Find the 1st term and the sum of the first ten positive terms. $Ans(-9,\ 10)$
5. The sum of the first five terms of an AP is $65/2$. Also, five times the 7th term is the same as six times the 2nd term. Find the first term and common difference of the AP. $Ans(a = 6, d = 1/4)$
6. The sum of the first ten terms of an AP is 95, and the sum of the first 20 terms of the same AP is 290. Calculate the 1st term and the common difference. $Ans(5,\ 1)$
7. Given that both the sum of the first ten terms of an AP and the sum of the 11th and 12th terms of the same AP are equal to 60, find the 1st term and the common difference. $Ans(-12,\ 4)$
8. The 17th term of an AP is 22, and the sum of the first 17 terms is 102. Find the 1st term, the common difference and the sum of the first 30 terms. $Ans(-10, 2,\ 570)$
9. An AP has 1st term 2 and common difference 5. Given that the sum of the first n terms of the progression is 119. Calculate the value of n. $Ans(7)$
10. Find how many terms of the AP $3 + 8 + 13 + \cdots$ should be taken in order that the total should exceed 200. $Ans(9)$
11. An AP has a common difference of 3. Given that the nth term is 32, and the sum of the first n terms is 185, calculate the value of n. $Ans(10)$
12. The 1st , 2nd and 3rd terms of an AP are $8 - x, 3x\ and\ 4x + 1$, respectively. Calculate the value of x, and find the sum of the first eight terms of the progression. $Ans(3, 152)$
13. Given that the 2nd, 3rd and 4th terms of an AP are $16 - x, 3x - 2\ and\ 2x$, respectively. Calculate the value of x, and state the 1st term of the progression. $Ans(4, 14)$
14. (a) Find the sum of the integers from 1 to 100
    (b) Find the sum of the integers from 1 to 100 which are divisible by 3
    (c) Hence find the sum of the integers from 1 to 100 which are not divisible by 3.
    $Ans(5050, 1683, 3367)$
15. Find the sum of the integers from 1 to 200 which are not divisible by 5. $Ans(16000)$
16. In an AP the nth term is 11, the sum of the first n terms is 72, and the first term is $\frac{1}{n}$. Find the value of n. $Ans(n = 13)$
17. If the roots of the equation $2x^3 + 3x^2 + kx - 6 = 0$ are in an arithmetic progression find the value of k and hence solve the equation.
18. Find the condition that the roots of the equation $x^3 + ax^2 + bx + c = 0$ are in Arithmetic progression.

10.2 Geometric progression (GP)

- A series in which each term is obtained from the preceding one by multiplying (or dividing) by a constant quantity is called a Geometrical progression **(GP)** e.g.1, 2, 4, 8 …,
- The terms of a **GP** are **a, ar, ar², ar³,…**
- The ratio between each term and the preceding one is called the **common ratio (r)**
- When three quantities are in a **G.P** the middle one is called the **Geometrical mean** $\mathbf{Gm} = \sqrt{\mathbf{ab}}$ if a and b are the two terms.

- In the series a, ar, ar^2, ar^3, … the power of r in any term is one less than the number of the term in the series. Thus ar^2 is a 3rd term
- The last term or nth term of the series is given by $\mathbf{L = ar^{n-1}}$
- The sum of the first **n** terms are given by

  $\mathbf{S_n = \frac{a(1-r^n)}{(1-r)}}$ $\mathbf{for\ |r| < 1\ ; r = 0.5, -.025, etc}$

  $\mathbf{S_n = \frac{a(r^n-1)}{(r-1)}}$ $\mathbf{for\ |r| > 1 r = 1.5, -2, etc}$

  Proof

  The sum of the first n terms is

  $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ ……………………….(1)

  Multiplying throughout by r gives

  $rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$ ………………………(2)

  Subtracting (2) from (1) gives

  $S_n - rS_n = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^n)$

  $\Rightarrow S_n(1 - r) = a - ar^n$

  $\Rightarrow S_n = \frac{a(1-r^n)}{1-r} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \blacksquare$ ……………………(3)

  Multiplying both the numerator and the denominator of (3) by -1 gives

  $S_n = \frac{a(r^n-1)}{(r-1)}$

  Which is an alternative form

Examples 14

Find three numbers in G.P such that their sum is 39 and their product is 729

Solution

Let the numbers be a/r, a and ar (3, 9, 27)

The product of the terms gives

$\left(\frac{a}{r}\right)(a)(ar) = 729$

$\Rightarrow a^3 = 729;\ \ a = 9$

The sum of the terms gives

$\left(\frac{9}{r}\right) + (9) + (9r) = 39$

$\Rightarrow 3r^2 - 10r + 3 = 0$

$\Rightarrow 3r^2 - 9r - r + 3 = 0$

$\Rightarrow 3r(r - 3) - (r - 3) = 0$

$\Rightarrow (3r - 1)(r - 3) = 0$

$\Rightarrow r = 3,\ r = 1/3$

When $r = 3$

The terms are $\left(\frac{9}{3}\right), (9), (9 \times 3) = 3, 9, 27$

When $r = 1/3$

The terms are 27, 9, 3

Example 15

Find the sum of ten terms of the GP: 2, -4, 8 . . .

Solution

$a = 2,\ r = \frac{-4}{2} = -2,\ n = 10$

Using $S_n = \frac{a(r^n-1)}{(r-1)}$

$= \frac{2((-2)^{10}-1)}{-2-1}$

$= -682$

Example 16

A G.P has first term 10 and common ratio 1.5, how many terms of the series are needed to reach a sum greater than 200?

Solution

$a = 10, r = 1.5$

Using $S_n = \frac{a(r^n-1)}{(r-1)}$

$\Rightarrow \frac{10((1.5)^n-1)}{1.5-1} = 200$

$\Rightarrow (1.5)^n = 11$

Taking log on both sides gives

$\log(1.5)^n = \log 11$

$\Rightarrow n = \frac{\log 11}{\log 1.5}$

$= 5.9139$

Thus for $S_n > 200$, we require $n > 5.9139$, i.e $n = 6$

Therefore the number of terms required to make a total greater than 200 is 6.

Example 17

Find the geometrical mean of the following 3 and 27

Solution

Let $a = 3$ and $c = 27$

Geometrical mean = $\sqrt{ac} = \sqrt{3 \times 27} = 9$

Example 18

Insert three geometric means between 162 and 1250

Solution

Let the terms be 162, p, q, z, 1250

Here we use the first term as 162 and the last term as 1250, then $n = 5$.

Using $L = ar^{n-1}$

$\Rightarrow 162r^4 = 1250;\ \ r = 5/3$

$\Rightarrow p = ar = 162\left(\frac{5}{3}\right) = 270$

$\Rightarrow q = ar^2 = 162\left(\frac{5}{3}\right)^2 = 450$

$\Rightarrow z = ar^3 = 162\left(\frac{5}{3}\right)^3 = 750$

The required geometrical means are 270, 450 and 750.

Example 19

A GP has a 1st term of 1 and a common ratio of $1/4$. Find the sum of the first four terms and show that the nth term is given by $4^{(1-n)}$.

Solution

The sum of the first four terms is given by

$S_4 = 1\left[\frac{1-\left(\frac{1}{4}\right)^4}{1-\frac{1}{4}}\right] = \frac{85}{64}$

The sum of the first four terms is $\frac{85}{64}$

The nth term is given by

$u_n = ar^{n-1}$

$= 1\left(\frac{1}{4}\right)^{n-1} = (4^{-1})^{n-1}$

$= 4^{(1-n)}$

Example 20

The sum of the 2nd and 3rd terms of a GP is 12. The sum of the 3rd and 4th terms is -36. Find the first term and the common ratio.

Solution

Since the sum of the 2nd and 3rd terms is 12, we have

$ar + ar^2 = 12$

$\Rightarrow ar(1+r) = 12$ ………………(1)

Since the sum of the 3rd and 4th terms is -36, we have

$ar^2 + ar^3 = -36$

$\Rightarrow ar^2(1+r) = -36$ ………………(2)

Dividing (2) by (1) gives

$\frac{ar^2(1+r)}{ar(1+r)} = \frac{-36}{12}$

$\Rightarrow r = -3$

From (1)

$\Rightarrow -3a(1-3) = 12; a = 2$

The 1st term of the GP is 2 and the common difference is -3.

Example 21

Show that there are two possible GPs in each of which the 1st term is 8 and the sum of the first three terms is 14. For the GP with positive common ratio find, in terms of n, an expression for the sum of the first n terms.

Solution

Since the 1st term is 8 and the sum of the first three terms is 14, we have

$8 + 8r + 8r^2 = 14$

$\Rightarrow 8r^2 + 8r - 6 = 0$

$\Rightarrow 2(2r-1)(2r+3) = 0$

Solving gives $r = 1/2 \ or \ r = -3/2$

Hence, there are two GPs which have a first term of 8 and have the sum of their three terms equal to 14, namely one with a common ratio of $1/2$ and a second with a common ratio of $-3/2$

To find the sum of the first n terms of the GP with positive common ratio, we use

$S_n = a\left(\frac{1-r^n}{1-r}\right)$ with $a = 8 \ and \ r = 1/2$

This gives

$$S_n = 8\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\left(\frac{1}{2}\right)}\right)$$

$$= 16[1-(2^{-1})^n]$$

$$= 16[1-2^{-n}]$$

Exercise 10.2

1. A GP has 3rd term 75 and 4th term 375. Find the common ratio and the first term. $Ans(5, 3)$
2. Find the sum of the first ten terms of a GP which has 3rd term 20 and 8th term 640. $Ans(5115)$
3. A GP has common ratio -3. Given that the sum of the first nine terms of the progression is 703, find the 1st term. $Ans(1/7)$
4. A GP has 1st term $\frac{1}{11}$ and common ratio 2. Given that the sum of the first n terms is 93. Calculate the values of n. $Ans(10)$
5. Find how many terms of the GP $5-10+20-\cdots$ should be taken in order that the total should equal 215. $Ans(7)$
6. In a geometric series the 1st term is 8 and the sum of the first three terms is 104. Calculate the possible values of the common ratio, and, in each case, write down the corresponding first three terms of the series.
   $Ans(r=-4; 8, -3, 128; r = 3; 8,\ 24, 72)$
7. In a GP the sum of the 2nd and 3rd tern is 12, and the sum of the 3rd and 4th terms is 60. Find the common ratio and the 1st term. $Ans\left(5, \frac{2}{5}\right)$
8. Find the first five terms in the geometric series which is such that the sum of the 1st and 3rd terms is 50, and the sum of the 2nd and 4th terms is 150.
   $Ans(5+15+45+135+405)$
9. In a GP in which all the terms are positive and increasing, the difference between the 7th and 5th terms is 192, and the difference between the 4th and 2nd is 24. Find the common ratio and the 1st term. $Ans(2, 4)$
10. A child tries to negotiate a new deal for her pocket money for the 30 days of the month of June. She wants to be paid 1p on the 1st of the month, 2p on the 2nd of the month, and, in general, $(2^{n-1})p$ on the nth day of the month, calculate how much she would get, in total, if this were accepted. $Ans(10737418.23)$
11. The 3rd,4th, and 5th terms of a GP are $x-2, x+1$ and $x+7$. Calculate the value of x and write down the first three terms of the progression. $Ans\left(5,\ \frac{3}{4}+\frac{3}{2}+3\right)$
12. A G.P has first term 16 and common ratio ¾. IF the sum of the first n terms is greater than 60. Find the least possible value of n.
13. The first term of a G.P is A and the sum of the first 3 terms is $\frac{7}{4}$A. Show that there are 2 possible progressions.
14. Solve the equation $54x^3-111x^2+74x-16=0$ given that the roots are ingeometric progression (answers: ½ , 2/3 , 8/9)

Mixed examples of AP and GP

Example 22

The 2^{nd} , 3^{rd} and 9^{th} term of an AP form a geometric progression. Find the common ratio of the GP.
Solution
The 2^{nd}, 3^{rd} and 9^{th} terms of an AP are given by $a + d, a + 2d\ and\ a + 8d$ respectively.
If these terms form a GP, the common ratio is given by
$r = \frac{a+2d}{a+d}\ or\ r = \frac{a+8d}{a+2d}$
Eliminating r gives
$\frac{a+2d}{a+d} = \frac{a+8d}{a+2d}$
$\Longrightarrow (a + 2d)^2 = (a + d)(a + 8d)$
$\Longrightarrow a^2 + 4ad + 4d^2 = a^2 + 9ad + 8d^2$
$\Longrightarrow 4d^2 + 5ad = 0$
$\Longrightarrow d(4d + 5a) = 0$
Solving gives $d = 0\ or\ d = -\frac{5a}{4}$
When $d = 0$, all the terms in the AP are the same and
$r = \frac{a+2(0)}{a+0} = 1$
In other words, all the terms in the GP are also the same.
When $d = -\frac{5}{4}a$
$r = \frac{a+2\left(-\frac{5}{4}a\right)}{a+\left(-\frac{5}{4}a\right)} = \frac{a-\frac{5}{2}a}{a-\frac{5}{4}a}$
$r = \frac{\left(-\frac{3}{2}\right)}{\left(-\frac{1}{4}\right)} = 6$
The common ratio of the GP is 6.

Sum to infinity

- Consider the general GP; $a+ar+ar^3+\ldots$
  $\mathbf{s_n} = \frac{\mathbf{a(1-r^n)}}{\mathbf{1-r}} = \frac{\mathbf{a}}{\mathbf{1-r}} - \frac{\mathbf{ar^n}}{\mathbf{1-r}}$
  If $-1<r<1$ then $\mathbf{s}_\infty = \frac{\mathbf{a}}{\mathbf{1-r}}$ this is the formula for sum to infinity.

Example 23
Find the sum to infinity of the following series 16+12+9+…
Solution
$a = 16, r = \frac{12}{16} = \frac{3}{4}$
$S_\infty = \frac{a}{1-r}$
$= \frac{16}{1-\frac{3}{4}}$
$= 64$

Example 24
Write the recurring decimal 0.3232 … as the sum of a GP. Hence write this recurring decimal as a rational number.
Solution
Now
$0.3232 \ldots = \frac{32}{100} + \frac{32}{10000} + \frac{32}{1000000} + \cdots$

This is a GP with $a = \frac{32}{100}$ and $r = \frac{1}{100}$. Since $-1 < r < 1$ the sum to infinity exists and is given by

$S_\infty = \frac{a}{1-r}$

$= \frac{\frac{32}{100}}{1-\frac{1}{100}}$

$= \frac{32}{99}$

The recurring decimal $0.\dot{3}\dot{2}$ can be written as $\frac{32}{99}$

Example 25

The sum to infinity of a GP is 7 and the sum of the first two terms is $\frac{48}{7}$. Show that the common ratio, r, satisfies the equation $1 - 49r^2 = 0$. Hence find the first term of the GP with positive common ratio.

Solution

Since the sum to infinity is 7, we have

$\frac{a}{1-r} = 7$

$\Longrightarrow a = 7(1 - r)$ ....(1)

The sum of the first two terms is $\frac{48}{7}$, therefore,

$a + ar = \frac{48}{7}$

$\Longrightarrow a(1 + r) = \frac{48}{7}$ ....(2)

Putting (1) into (2)

$\Longrightarrow 7(1 - r)(1 + r) = \frac{48}{7}$

$\Longrightarrow 49(1 - r^2) = 48$

$\therefore 1 - 49r^2 = 0$ ....■

Solving gives $r = 1/7$ $or = -1/7$

Since we require the GP with positive common ratio $r = 1/7$, from (1) the first term is given by

$a = 7(1 - r)$

$= 7\left(1 - \frac{1}{7}\right)$

$a = 6$

The first term of the GP with positive common ratio is 6.

The compound interest

- Now after n years the amount will be given by $\mathbf{A = p\left(1 + \frac{r}{100}\right)^n}$

  Example 26

  Mr. Lule deposits shs.150000/= in a bank at the beginning of every year for 11 years. How much money does he receive at the end of this period if he is paid a compound interest of 12.5% per annum? (3581868.80/=)

  Solution

  Using $\mathrm{A = p}\left(1 + \frac{\mathrm{r}}{100}\right)^{\mathrm{n}}$ where $p = \frac{150000}{=}$, $r = 12.5\%, n = 11$

  Amount after the 1st year $A_1 = 150000\left(1 + \frac{12.5}{100}\right)^1 = 150000(1.125)$

  Amount after the 2nd year $A_2 = 150000(1.125)^2$

Amount after the 3rd year $A_2 = 150000(1.125)^3$

.

.

.

Amount after the 11th year $A_{11} = 150000(1.125)^{11}$
The total amount $= A_1 + A_2 + A_3 + \cdots + A_{11}$
$= 150000(1.125) + 150000(1.125)^2 + 150000(1.125)^3 + \cdots + 150000(1.125)^{11}$
$= 150000(1.125)\{1 + 1.125 + (1.125)^2 + \cdots + (1.125)^{10}\}$
$= 150000(1.125)\{\text{sum of a GP}\}$; with $a = 1, r = 1.125, n = 11$
$= 150000(1.125)\left[1\left(\frac{(1.125)^{11}-1}{1.125-1}\right)\right]$
$= 3581868.807/=$
Lule receives 3581868.807/= after 11 years

10.3 Proof by induction

- This involves simplifying one side of the equation to resemble the original form but passing through various steps.
- The following are the steps
  i) Try n=1 and see if it holds
  ii) Say suppose it holds for n=k
  iii) Then finally add a next term on the LHS and simplify the expression.

Examples 27

Prove by induction that for all positive integral values of n,
$1 + 2 + 3 + 4 + \cdots + n = \frac{n}{2}(n + 1)$

Solution

For n=1
L.H.S gives 1; R.H.S gives $\frac{1}{2}(1 + 1)$=1, hence it holds for $n = 1$
Suppose it holds for $n = k$, such that $1 + 2 + 3 + 4 + \cdots + k = \frac{k}{2}(k + 1)$
Adding the next term $n = (k + 1)$ such that;
$1 + 2 + 3 + 4 + \cdots + k + (k + 1) = \frac{k}{2}(k + 1) + (k + 1)$
$=(k + 1)\left[\frac{k}{2} + 1\right]$
$=\frac{(k+1)}{2}[k + 2]$
But $k = n - 1$
$=\frac{(n-1+1)}{2}[n - 1 + 2]$
$=\frac{n}{2}[n + 1]$
Since it holds for n=1, k and k+1, it holds for other integral values of n such as 2, 3, 4...
$\therefore \mathbf{1 + 2 + 3 + 4 + \cdots + n = \frac{n}{2}(n + 1)}$

A more concise way of writing the series $1 + 2 + 3 + 4 + \cdots + n$ is to use the symbol Σ (a Greek letter pronounced 'sigma')
$\sum_{r=1}^{n} f(r)$ means the series with first term $f(1)$, the second term $f(2)$,... last term $f(n)$

Thus $1 + 2 + 3 + 4 + \cdots + n = \sum_{r=1}^{n} r$ and the result proved above can be written as $\sum_{r=1}^{n} r = \frac{n}{2}(n + 1)$

Example 28

Prove by mathematical induction that $\sum_{\mathrm{r=1}}^{\mathrm{n}} \frac{1}{\mathrm{r(r+1)}} = \frac{\mathrm{n}}{\mathrm{n+1}}$

Solution

If $n = 1$ LHS= $\sum_{\mathrm{r=1}}^{1} \frac{1}{\mathrm{r(r+1)}} = \frac{1}{1(1+1)}$; $\quad RHS = \frac{1}{1+1} = \frac{1}{2}$

$= \frac{1}{2}$

Thus the statement is true for $n = 1$

Assume the statement is true for $n = k$ such that;

$\sum_{\mathrm{r=1}}^{\mathrm{k}} \frac{1}{\mathrm{r(r+1)}} = \frac{\mathrm{k}}{\mathrm{(k+1)}}$

Adding the next term $n = k + 1$; then $\sum_{\mathrm{r=1}}^{\mathrm{k+1}} \frac{1}{\mathrm{r(r+1)}} = \sum_{\mathrm{r=1}}^{\mathrm{k}} \frac{1}{\mathrm{r(r+1)}} + \frac{1}{\mathrm{(k+1)(k+2)}}$

$= \frac{\mathrm{k}}{\mathrm{(k+1)}} + \frac{1}{\mathrm{(k+1)(k+2)}}$

$= \frac{k(k+2)+1}{(k+1)(k+2)}$

$= \frac{k^2+2k+1}{(k+1)(k+2)}$

$= \frac{k^2+k+k+1}{(k+1)(k+2)}$

$= \frac{(k+1)^2}{(k+1)(k+2)}$

$= \frac{(k+1)}{(k+2)}$

$= \frac{(k+1)}{(k+1+1)}$

$= \frac{n}{(n+1)}$

Since it holds for $n = 1, k, k + 1$, then it is true for other values of n such as 2, 3, 4 …

Thus it is true that $\sum_{\mathrm{r=1}}^{\mathrm{n}} \frac{1}{\mathrm{r(r+1)}} = \frac{\mathrm{n}}{\mathrm{n+1}}$

Example 29

Show that $9^n - 1$ is a multiple of 8 for all positive values of n.

Solution

For n=1; $9^1 - 1 = 8$ which holds for n=1

Suppose it holds for $n = k$ such that $9^{\mathrm{k}} - 1 = 8\mathrm{p}$; $9^{\mathrm{k}} = 8\mathrm{p} + 1$

Adding another term $n = k + 1$;

$\Rightarrow 9^{\mathrm{k}} - 1 = 9(9^{\mathrm{k}}) - 1$

$= 9(8\mathrm{p} + 1) - 1$

$= 72\mathrm{p} + 8$

$= 8(9\mathrm{p} + 1)$

$= 8\mathrm{N}$ where N is a real number.

Since it holds for $n = 1, k$ and $k + 1$, then it will hold for other values of n such as 2, 3 and so on. Hence $9^{\mathrm{n}} - 1$ is a multiple of 8 for all positive values of n.

Example 30

Prove by induction that $10^n + 3 \times 4^{n+2} + 5$ is a multiple of 9 for $n \geq 1$

Solution

For $n = 1, 10^n + 3 \times 4^{n+2} + 5 = 10^1 + 3 \times 4^{1+2} + 5 = 207$

$= 9(23)$

Suppose it hold s for $n = k$, such that; $10^k + 3 \times 4^{k+2} + 5 = 9A$ where A is a whole number.

For $n = k + 1$;

$\Longrightarrow 10^{k+1} + 3 \times 4^{k+3} + 5 = 10(10^k) + (3 \times 4^{k+3}) + 5$ ....(1)

From $10^k + 3 \times 4^{k+2} + 5 = 9A$; $10^k = 9A - (3 \times 4^{k+2} + 5)$.................(*)

Putting equation (*) into (1)

$$\Longrightarrow 10^{k+1} + 3 \times 4^{k+3} + 5 = 10\left(9A - (3 \times 4^{k+2} + 5)\right) + (3 \times 4^{k+3}) + 5$$
$$= 90A - 30 \times 4^{k+2} - 45 + 3 \times 4 \times 4^{k+2}$$
$$= 90A - 18 \times 4^{k+2} - 45$$
$$= 9(10A - 2 \times 4^{k+2} - 5)$$

$= 9N$ where N is an integer

$\therefore$ Since it holds for $n = 1, k \text{ and } k + 1$, then it will hold for other values of n such as 2, 3 and so on. Hence $10^n + 3 \times 4^{n+2} + 5$ is a multiple of 9 for $n \geq 1$.

Exercise 10.3

1. In a G.P the sum of the second and third terms is 12. The sixth term is 9 times the fourth term. Find the 1st three terms of the G.P if it has only positive terms.
2. The age of John and his three children are in G.P the sum of their ages is 120years, and the sum of the ages of the two young children is 12 years. Find Kato's age
3. (a) How many terms are there in the A.P $4\frac{1}{2} + 7 + \cdots + 32$? Find the sum of the A.P

   (b)The sum of the first two terms of the G.P is -8 and the sum of the fourth and fifth terms of the G.P is 216. Find the 1st term and common ratio. Calculate the sum of the 1st eight terms.
4. Find two distinct numbers p and q such that p,q,10 are in A.P and q, p, 10 are in G.P (-5, 2.5)
5. a) The first term of an A.P is 73 and the ninth term is 25, Determine

   i) The common difference of the A.P

   ii) The number of terms that must be added to give the sum of 96

   b)A G.P and an A.P has the same 1st term. The sums of their 1st ,2nd and 3rd terms are 6, 10.5 and 18 respectively. Calculate the sum of their fifth terms.
6. (a) The 10th term of an A.P is 69 and the sum of the 1st 30 terms is 4 times the sum of the 1st ten terms. Find the common difference and the first term of the A.P

   (b)The angles of a triangle form a G.P. If the smallest is 20^0, Determine the largest angle correct to the nearest degree.
7. How many terms of the G.P $\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \cdots$ are needed to make a total of $2^{16} - \frac{1}{16}$
8. Show that if $\log a$, $\log b$ and $\log c$ are consecutive terms of a A.P then a, b and c are consecutive terms of a G.P
9. Find the condition that the roots of the equation $x^3 + ax^2 + bx + c = 0$are in Geometric progression.

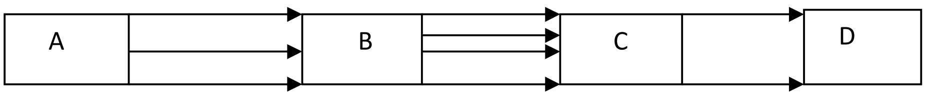
10. Jack operates an account with a certain bank which pays a compound interest of 13.5% per annum. He opened the account at the beginning of the year with 500000/= and deposits the same amount of money at the beginning every year.
a) Calculate how much money he will receive at the end of 9years.
b) After how long the money have accumulated to 3.32millions.
11. A credit society gives out a compound interest of 4.5% per annum. Mugogo deposits shs.300000/= at the beginning of each year. How much money will he have at the beginning of 4 years if there are no withdrawals during this period? (1341212.917/=)
12. Prove by induction that the sum of the series $1^2 + 2^2 + 3^2 + \cdots + n^2$ is $\frac{n}{6}(n+1)(2n+1)$
13. Prove by induction the following expressions
$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2}{4}(n+1)^2$
14. Use the method of induction to prove that $\sum_{r=1}^{n} 2^{r-1} = 2^n - 1$
15. Use the method of induction to prove that $6^n - 1$ is divisible by 5 for all positive integral values of n
16. Prove that $8^n - 7n + 6$ is a multiple of 7 for all positive integral values of n
17. Prove by induction that $\mathbf{f(n) = 3^{2n+2} - 8^n - 9}$ is a factor of 64.
18. Prove by induction that $\sum_{r=1}^{n} \mathbf{r^2(r+1) = \frac{n}{12}(n+1)(n+2)(3n+1)}$
19. Show that $\mathbf{1.2 + 2.3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}}$
20. Show that $\sum_{2}^{n+1} \mathbf{r.2^{r-2} = n.2^n}$
21. Prove that $\mathbf{2^{n+2} + 3^{2n+1}}$ is a multiple of 7.
22. Prove that $\mathbf{6^n + 8^n}$ is a multiple of 7 if n is odd.
23. Show that $\mathbf{6^n - 8^n}$ is a multiple of 7 for all even values of n.
24. Show that $\mathbf{n^3 - n}$ is a multiple of 6.
25. Prove by induction that $\mathbf{17^n - 1}$ is a multiple of 16.
26. Show that $\mathbf{5^{6n} + 2^{3n+1} - 3}$ is a multiple of 7 for all positive integral values of n.
27. Show that $\mathbf{5^{2n} - 3^{2n}}$ is a multiple of 8 for all positive integral values of n.
28. Show that $\mathbf{3^{4n+3} + 53}$ is a multiple of 80 for all positive integral values of n.
29. Prove by induction that $\mathbf{f(n) = n^3 - n}$ is a factor of 6 for all integral values of n such that $n > 2$

CHAPTER 11

PERMUTATION AND COMBINATION

Successive operations

If there are 3 paths joining A to B and 4 path joining B to C, then there are (3x4) or 12 different ways of joining from A, through B to C.



Example 1

There are five roads joining A to B and 3 roads joining B to C. Find how many different routes there are from A to C via B

Solution
There are 5x3=15 routes

Example 2
A man has three choices of the way in which he travels to work; he can walk, go by car or go by train. In how many different ways can arrange his travel for the 5 working days in the week?
Solution
On Monday he has three choices, walk, car, train
On the rest of the days similarly Tue, Wed, Thus and Fri.
Hence there are 5 successive operations, each of which can be performed in 3 ways
Total number of arrangements=3x3x3x3x3=$3^5 = 243$ways.

Arrangements

- Consider the letters A, B and C. If these letters are written in a row one after another, and we can arrange these letters in form of 3 letters e.g. (ABC, BCA,…)
- A useful shorthand way of getting the number of arrangements is n! (Read as n factorial)
- Thus 4! =4x3x2x1, 5! =5x4x3x2x1 and so on.

Example 3
Evaluate a) $\frac{6!}{2\times4!}$ b) $\frac{7!}{4!\times2!}$
Solution
(a) $\frac{6!}{2\times4!} = \frac{6\times5\times4!}{2\times4!}$
$= 15$
(b) $\frac{7!}{4!\times2!} = \frac{7\times6\times5\times4!}{4!\times2!}$
$= \frac{210}{2}$
$= 105$

Example 4
Five children are to be seated on a bench find;
(a) How many ways the children can be seated.
(b) How many arrangements are possible if the youngest child is to sit at the left-hand end of the bench?
Solution
(a) $\dot{5}\ \dot{4}\ \dot{3}\ \dot{2}\ \dot{1}$=5!=120 ways
(b) $1 \times 4! = 24 ways$

Example 5
Three different mathematics books and other 5 different books are to be arranged on a bookshelf. Find
(a) The number of possible arrangements of the books
(b) The number of possible arrangements if the three mathematics books must be kept together.
Solution
a) $8! = 40320$ways.

b) Since the 3 mathematics books are to be together, consider these bound together as one book. There are now 6books to be arranged and this can be performed in 6! =720ways.
Now in each of these arrangements the 3 mathematics books are bound together; these mathematics books can be arranged in 3x2x1=3! Ways.
=6ways
The total number of arrangements =720x6=4320ways

Circular arrangements

- The number of arrangements of n unlike things in a circle is given by **(n-1)!**
- In those cases where clockwise and anticlockwise arrangements are not considered to be different this reduces to $\frac{1}{2}(\mathbf{n}-\mathbf{1})!$

Example 6

Four men Peters, Rogers, Smith and Thomas are to be seated at a circular table, in how many ways can this be done?
Solution
Number of ways= (4-1)! Ways

Example 7

Nine beads all of different colours are to be arranged on a circular wire, two arrangements are considered not to be different as they appear the same when the ring is turned over. How many different arrangements are possible?
Solution

$$\text{Number of ways} = \frac{1}{2}(n-1)!$$
$$= \frac{1}{2}(9-1)!$$
$$= 20160 ways$$

Mutually exclusive situations

- When two situations A and B are mutually exclusive then, if situation A occurs, situation B cannot occur. And vice-versa.
- In such cases the number of arrangements of either situation A or situation B occurring can be obtained by adding the number of arrangements of situation A to the number of arrangement of situation B

Example 8

How many different four digit numbers can be formed from the figures 3, 4, 5, 6 if each figure is used only once in each number? How many of the numbers?

a) end in a 4
b) end in a 3
c) end in a 3 or a 4?

Solution
The 1st digit can be chosen in 4ways, 2nd in 3ways and so on.
Thus there are 4x3x2x1=24 different four digit numbers.

a) The last digit can be chosen in 1 way as it must be a 4, the first digit can be chosen in 3ways, the second in 2ways and the third 1way.
Thus there is 1x3x2x1=6 of the numbers that end in a 4

b) By reasoning similarly there will be 1x3x2x1=6 of the numbers that end in a 3
c) The number that end in a 3 cannot end in a 4, so these are mutually exclusive events =6+6=12 of the numbers end either in a 3 or a 4.or 2x3x2x1=12ways.

NB the permutations in which a certain event A does not occur will clearly be mutually exclusive with those permutations in which that event does not occur.

$$\begin{Bmatrix} No\ of\ permutations \\ in\ which\ event\ A \\ does\ occur \end{Bmatrix} + \begin{Bmatrix} No\ of\ permutations \\ in\ which\ event\ B \\ does\ occur \end{Bmatrix} = \begin{Bmatrix} No\ of\ permutations\ in \\ which\ either\ event\ A \\ does\ or\ doesnot\ occur \end{Bmatrix}$$
$$= total\ number\ of\ permutaions.$$

Example 9

In how many ways can five people Smith, Jones, Clerk, Brown and White be arranged around a circular table if

a) Smith must sit next to Brown
b) Smith must not sit next to Brown.

Solution

a) Since Smith and Brown must sit next to each, consider these two bound together as one person. There are now 4 people to sit. Fix one of these and then the remaining 3 can be seated in 3x2x1=6ways relative to the one that was fixed.
In each of these arrangements Brown and Smith are seated together in a particular way. Brown and Smith could now change seats giving another 6ways of arranging the 5 people. The total no of arrangements=2x6=12ways
b) If smith and Brown are not to sit together the this situation is mutually exclusive with the situation in (a)
Total number of ways $=(n-1)$
Total number $=(5-1)!=4!$
Required number of arrangements=4!-12
=12ways

11.1 Permutation of objects selected from a group

- We usually say that the number of permutations of **r** objects selected from **n** unlike objects is $\mathbf{n_{p_r}}$ where $\mathbf{n_{pr}=\frac{n!}{(n-r)!}}$
- Thus the case of arranging two letters chosen from the five letters A, B, C, D and E mentioned above, this gives; $5p_2=\frac{5!}{3!}=20$ as required.

NB. $\mathbf{np_n=\frac{n!}{(n-n)!}=n!}$ since **0 !=1**

Example 10

Ten athletes are to take part in a race. In how many different ways can the 1^{st},$2^{nd}$and 3^{rd} place be filled?

1^{st} Place to be filled in 10 ways
2^{nd} Place can be filled in 9 ways
3^{rd} Place can be filled in 8 ways
Total number of permutations of 1^{st} ,2^{nd},and, 3^{rd}
=10x9x8=720.
Or np_r= 10p_3=$\frac{10!}{7!}$

=720

Arrangements of like and unlike things

- The number of ways of arranging n objects where p are like q are like and r are also alike is given by $\frac{\mathbf{n}!}{\mathbf{p}!\mathbf{q}!\mathbf{r}!}$.

Example 11

In how many ways can 4red, 3yellow and 2green discs be arranged in a row, if discs of the same colour are indistinguishable?

Solution

4+3+2=9 discs

The number of arrangements $=\frac{9!}{4!3!2!}$=1260ways.

Example 12

Find

a) in how many different ways the letters of the word ALGEBRA can be arranged in a row

b) in how many of these arrangements the two A's are together,

c) in how many of the arrangement the two A's are not together.

Solution

a)$No\ of\ arrangement = \frac{7!}{2!}$=<u>2520</u>

b) If the A's are kept together there are effectively 6 letters to arrange, hence number of arrangements =6! =720.

(c) No of ways =2520-720 when A's not together.

=1800 ways.

Example 13

In how many ways can the letters of the word **MATHEMATICS** be arranged in a row.

Solution

There are 2M, 2T, 2A and the total number of letters is 11.

$No\ of\ ways = \frac{11!}{2!2!2!}$

$= 4989600$

11.2 Combinations

- A combination is a selection of r objects from n objects and the order is not important.
- Thus the number of possible combinations of n different objects, taken at a time is given by $\mathbf{nc_r}$ also written as $\binom{\mathbf{n}}{\mathbf{r}}$

Where $\mathbf{nc_r} = \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{r})!\mathbf{r}!}$

Example 14

How many selections of 4 letters can be made from the 6 letters .a, b, c, d, e and f?

Solution

The number of selections is nc_r where r is the number of things selected from a group of n .Hence for this case n=6 and r=4.

$6c_4=\frac{6!}{2!4!}=15$

There are 15 selections of 4 letters which can be made from the 6 letters.

Example 15

How many different commutes, each consisting 3 boys and 2 girls can be chosen from 7 boys and 3 girls?

Solution

7 boys	3 girls
3	2

Number of ways of choosing 3 boys from 7

$=7c_3 \quad =\frac{7!}{4!3}$ =35

Number of ways of choosing 2 girls from 5

$=5c_2 = \frac{5!}{2!2!}$ =10

Number of commutes which can be chosen

=35x 10=350

Example 16

A team of 7 players is to be chosen from a group of 12 players, one of the 7 is then to be selected as captain and another as Vice-captain .In how many ways can this be done

Number of ways of choosing 7 from 12 is $12c_7$

$=12c_7 =\frac{12!}{5!7!}$

=792

In each choice there are 7 ways of choosing a captain and there are 6 ways of choosing a captain.

(Alternatively we say the captain and vice-captain can be elected in $7p_2$ ways)

Total number of selections is $7 \times 6 \times 792 = \ 33264\ ways$

Example 17

A group consists of 4 boys and 7 girls, in how many ways can a team of five be selected if it is to contain

(a) No boys
(b) At least one number of each sex
(c) 2 boys and 3 girls.
(d) At least 3 boys.

Solution

(a) No boy is selected so the team is chosen from the 7 girls

Number of ways of choosing 5girls from 7 is $7c_5 =\frac{7!}{2!5!} = 21$

(b) Considering mutually exclusive groups

Number of selections with no boys = 21.

Number of selections with no girls = 0 (as there are only 4 boys)

Total number of possible selections $=11c_5$=462

The number of selections with at least one of each sex is =462-21=441

(c) 2 boys can be chosen from 4 ,in $4c_2$=6 ways

3 girls can be chosen from 7 in $7c_3 = 35$ ways
These are independent events so
Number of events with 2 boys and 3 girls are =6x 35
=210

(d) If the team is to have at least 3 boys, then there must be either 3 or 4 boys.
Number of teams with 3 boys and 2 girls =$4c_3$ x$7c_2$ =84
Number of teams with 4 boys and 1 girl =$4c_4$ x $7c_1$ =7
Number of teams with at least 3 boys =84+ 7
=91

Exercise 11a

1. Evaluate the following (a) $\frac{8!}{6!}$ (b) $\frac{9!}{3\times5!}$ (c) $\frac{5!\times4!}{6!}$ $Ans((a)56\ (b)1008\ \ (c)4)$
2. Express the following in terms of 5!
(a) $\frac{6!}{6}$ (b) $6! - 5!$ (c) $2 \times 6! - 3 \times 5!$ $Ans\big((a)5!\,(b)5 \times 5!\,(c)9 \times 5!\big)$
3. There are three roads joining town X to town Y; three more roads join Y to Z and two roads join Z to A. How many different routes are there from X to A passing though Y and Z? $Ans(18)$
4. In how many ways can a group of 10 children be arranged in a line? $Ans(10!)$
5. The front doors of five houses in a terrace are to be painted blue, brown black, green and red. In how many different ways can the painting be done if no two doors are to be the same colour? $Ans(120)$
6. In how many different ways can the letters of the word THURSDAY be arranged? $Ans(8!)$
7. The letters of the word TUESDAY are arranged in a line, each arrangement ending with the letter S. How many of these arrangements also start with the letter D? $Ans(720, 120)$
8. How many ways can five women be seated at a circular table? $Ans(24)$
9. How many numbers greater than 50,000 can be formed using the digits 2, 3, 4, 5 and 6 if each number is used only once in each number? $Ans(48)$
10. There are nine different books on a shelf, one of which is a dictionary and one an atlas. In how many ways can the books be arranged on the shelf if the dictionary and the atlas are to be next to each other? $Ans(80640)$

Exercise 11b

1. Find the number of permutations of two different letters taken from the letters A, B, C, D, E and F. $Ans(30)$
2. In how many ways can six books be arranged on a shelf when the books are selected from 10 different books? $Ans(151200)$
3. How many three-digit numbers, with all three digits different, can be formed from the figures 7, 6, 5, 4, 3 and 2? $Ans(120)$
4. How many different teams of 7 payers can be chosen from 10 girls? $Ans(120)$
5. How many different hands of three cards can be chosen from a pack of 52 cards? $Ans(22100)$
6. A group consists of 5 boys and 8 girls. In how many ways can a team of four be chosen, if the team contains
(a) No girl (b) no more than one girl (c) at least two boys
$Ans\big((a)5\ (b)85(c)365\big)$

7. In how many ways may a committee of 5 people be selected from 7 men and 3 women , if it must contain
   (a) 3 men and 2 women (b) 3 women and 2 men (c) at least 1 women.
   $Ans\big((a)105(b)21(c)231\big)$
8. In how many ways can a committee of 7 people be selected from 4 men and 6 women if the committee must have at least4 women on it? $Ans(100)$
9. A committee of 4men and 3women is to be formed from 10 men and 8 women. In how many ways can the committee be formed?
10. A group of nine has to be selected from ten boys and eight girls. It can consist of either five boys and four girls or four boys and five girls. How many different groups can be formed?

CHPATER 12

BINOMIAL THEOREM

Pascal's triangle

By expanding it can be shown that

$(a + x)^1 = a + x$

$(a + x)^2 = a^2 + 2ax + x^2$

$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$

$(a + x)^4 = a^4 + 4a^3x + 6a^2x + 4ax^3 + x^4$

- Extracting the coefficient of a and x are obtained in a **Pascal triangle**

1 1
1 2 1
1 3 3 1
1 4 6 4 1

Pascal's triangle can be used to expand an expression of the type $(a + x)^n$.

Example 1

Expand $(2 - x)^4$ using Pascal's triangle

Solution

$$\Rightarrow (2 - x)^6 = 1(2)^4(-x)^0 + 4(2)^3(-x)^1 + 6(2)^2(-x)^2 + 4(2)^1(-x)^3 + 1(2)^0(-x)^4$$
$$= 16 - 32x + 24x^2 - 8x^3 + x^4$$

Example 2

Expand $(5 + 2x)^3$ using Pascal's triangle

Solution

$$\Rightarrow (5 + 2x)^3 = 1(5)^3(2x)^0 + 3(5)^2(2x)^1 + 3(5)^1(2x)^2 + 1(5)^0(2x)^3$$
$$= 125 + 150x + 60x^2 + 8x^3$$

Example 3

Expand $(x - \frac{1}{x})^4$ using Pascal's triangle

Solution

$$\Rightarrow \left(x - \frac{1}{x}\right)^4 = 1(x)^4\left(\frac{1}{x}\right)^0 + 4(x)^3\left(\frac{1}{x}\right)^1 + 6(x)^2\left(\frac{1}{x}\right)^2 + 4(x)^1\left(\frac{1}{x}\right)^3 + 1(x)^0\left(\frac{1}{x}\right)^4$$
$$= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$$

NB. The **Pascal's triangle** cannot be used for large values of n. so we use the **binomial theorem.**

The binomial theorem for a positive integral index

- Generally, we denote the entry in the nth row, $(r+1)th$ position by $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ where $n! = n(n-1)(n-2)\ldots 3.2.1$ $(called\ n\ factorial)$ and where by definition $0! = 1$.
- Using this definition we can write a general formula for the expansion of $(1+x)^n$ $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + x^n$
- This is known as the binomial expansion of $(1+x)^n$,, for positive integer n. This can also be written using sigma notation $(1+x)^n = \sum_{r=0}^{n}\binom{n}{r}x^r$
- Alternatively the expansion can be written as; $(1+x)^n=1+\frac{nx}{1!}+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\ldots\ldots+x^n$
- Notice that this expansion terminates at the term x^n when n is a positive integer.

Example 4

Evaluate the following;

(a) $\binom{5}{1}$ (b) $\binom{6}{4}$

Solution

(a) $\binom{5}{1} = \frac{5!}{(5-1)!1!}$

$= \frac{5!}{4!1!}$

$= \frac{5.4.3.2.1}{4.3.2.1.1}$

$= 5$

(b) $\binom{6}{4} = \frac{6!}{(6-4)!4!}$

$= \frac{6!}{2!4!}$

$= \frac{6.5.4.3.2.1}{2.1.4.3\times 2.1}$

$= 15$

Example 5

Find the coefficient of the x^2 and x^3 terms in the expansion of $(1+x)^7$

Solution

The coefficient of x^2 term is

$\binom{7}{2} = \frac{7!}{(7-2)!2!} = 21$

The coefficient of x^2 term in the expansion of $(1+x)^7$ is 21

The coefficient of x^3 term is

$\binom{7}{3} = \frac{7!}{(7-3)!3!} = 35$

The coefficient of x^3 term in the expansion of $(1+x)^7$ is 35.

Example 6

Expand $(3+y)^4$ in powers of y.

Solution

In order to use the binomial expansion, we need to express $(3+y)^4$ in the form $(1+x)^4$. Now

$(3+y)^4 = \left[3\left(1+\frac{y}{3}\right)\right]^4 = 3^4\left(1+\frac{y}{3}\right)^4$

Hence we can use the binomial expansion of $(1+x)^4$ with $x=\frac{y}{3}$, which gives

$$
\begin{aligned}
(3+y)^4 &= 3^4\left(1+\frac{y}{3}\right)^4 \\
&= 3^4\left[1+\binom{4}{1}\left(\frac{y}{3}\right)+\binom{4}{2}\left(\frac{y}{3}\right)^2+\binom{4}{3}\left(\frac{y}{3}\right)^3+\left(\frac{y}{3}\right)^4\right] \\
&= 81\left[1+4\left(\frac{y}{3}\right)+6\left(\frac{y}{3}\right)^2+4\left(\frac{y}{3}\right)^3+\left(\frac{y}{3}\right)^4\right] \\
&= 81\left(1+\frac{4y}{3}+\frac{6y^2}{9}+\frac{4y^3}{27}+\frac{y^4}{81}\right)
\end{aligned}
$$

$\therefore (3+y)^4 = 81+108y+54y^2+12y^3+y^4$

Example 7

Expand (1+x+x^2)$(1-x)^8$ in ascending powers of x up to and including the terms in x^3

Solution

Using the expansion $(1+x)^n$ with $n=8$ and $x=(-x)$

$(1-x)^8$ =1-8x +28x^2-56x^3+…….

(1+x+x^2)$(1-x)^8$ = (1+x+x^2) (1-8x +28x^2-56x^3+…)

Expanding and ignoring terms of order x^4 and above gives

(1+x+x^2)$(1-x)^8 = 1-7x+21x^2 \ -36x^3$ …

Example 8

Given that $(1-2x)^5(2+x)^6 \equiv a+bx+cx^2+dx^3+\cdots$, find the values of the constants a, b, c and d

Solution

Expanding $(1-2x)^5$ using the binomial theorem gives;

$$
\begin{aligned}
(1-2x)^5 &= 1+\binom{5}{1}(-2x)+\binom{5}{2}(-2x)^2+\binom{5}{3}(-2x)^3+\cdots \\
&= 1-10x+40x^2-180x^3+\cdots
\end{aligned}
$$

Expanding $(2+x)^6$ using the binomial theorem gives;

$$
\begin{aligned}
(2+x)^6 = 2^6\left(1+\frac{x}{2}\right)^6 &= 2^6\left[1+\binom{6}{1}\left(\frac{x}{2}\right)+\binom{6}{2}\left(\frac{x}{2}\right)^2+\binom{6}{3}\left(\frac{x}{2}\right)^3+\cdots\right] \\
&= 64\left[1+3x+\frac{15x^2}{4}+\frac{20x^3}{8}+\cdots\right] \\
&= 64+192x+240x^2160x^3+\cdots
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&(1-2x)^5(2+x)^6 \\
&\quad = (1-10x+40x^2-180x^3+\cdots)(64+192x+240x^2160x^3+\cdots) \\
&\quad = 64-448x+880x^2+320x^3+\cdots
\end{aligned}
$$

Therefore, the values of the constants a, b, c and d are

$a=64,\ b=-488,\ c=880,\ d=320$

Example 9
Write down the expansion of $(1+y)^4$. Hence find the first four terms in the expansion of $(1+x+x^2)^4$.
Solution
$$(1+y)^4 = 1+\binom{4}{1}y+\binom{4}{2}y^2+\binom{4}{3}y^3+y^4$$
$$= 1+4y+6y^2+4y^3+y^4 \text{ ....}(1)$$
Writing $(1+x+x^2)^4$ as $[1+(x+x^2)]^4$ and using (1) with $y = x+x^2$ give
$$[1+(x+x^2)]^4 = 1+4(x+x^2)+6(x+x^2)^2+4(x+x^2)^3+(x+x^2)^4$$
$$= 1+4(x+x^2)+6x^2(1+x)^2+4x^3(1+x)^3+x^4(1+x)^4$$
Since we require only the first four terms of the expansion, we can ignore terms involving x^4 and higher powers of x. Therefore,
$$[1+(x+x^2)]^4 = 1+4(x+x^2)+6x^2(1+2x)+4x^3(1)$$
$$\therefore (1+x+x^2)^4 = 1+4x+10x^2+16x^3+\cdots$$

Example 10
Expand $(1+x-x^2)^7$ in ascending powers of x up to and including the term in x^3.
Solution
Using $(1+x)^n$ with $n=7$ and $(x-x^2)$ substitute for x;
$$(1+x-x^2)^7 = [1+(x-x^2)]^7$$
$$= 1+\binom{7}{1}(x-x^2)+\binom{7}{2}(x-x^2)^2+\binom{7}{3}(x-x^2)^3+\cdots$$
$$= 1+7(x-x^2)+21x^2(1-x)^2+35x^3(1-x)^3+\cdots$$
$$= 1+7(x-x^2)+21x^2(1-2x)+35x^3(1)+\cdots$$
$$\therefore (1+x-x^2)^7 = 1+7x+14x^2-7x^3+\cdots$$

Example 11
Given that the first three terms in the expansion in ascending powers of x of $(1+x+x^2)^n$ are the same as the first three terms in the expansion of $\left(\frac{1+ax}{1-3ax}\right)^3$, find the value of a and n.
solution
$$(1+x+x^2)^n = 1+n(x+x^2)+\frac{n(n-1)(x+x^2)^2}{2!}+\ldots$$
$$= 1+nx+nx^2+\frac{1}{2}n(n-1)x^2+\ldots$$
$$\left(\frac{1+ax}{1-3ax}\right)^3 = (1+ax)^3(1-3ax)^{-3}$$
$$= (1+3ax+3a^2x^2+\ldots)(1+9ax+18a^2x^2+\ldots)$$
$$= 1+9ax+54a^2x^2+3ax+27a^2x^2+3a^2x^2+\ldots$$
$$= 1+12ax+84a^2x^2+\ldots$$
Thus $1+nx+nx^2+\frac{1}{2}n(n-1)x^2+\ldots = 1+12ax+84a^2x^2+\ldots$
Equating coefficients, we get $n = 12a$ ....(i)

$84a^2 = n + \frac{n(n-1)}{2}$ ....(ii)

$84a^2 = 12a + \frac{12a(12a-1)}{2}$, $84a^2 = 12a + 72a^2 - 6a$

$12a^2 - 6a = 0$, for $a \neq 0$, $a = \frac{1}{2}$ **thus** $n = 6$

Example 12

Calculate the value of the constant a if the coefficient of the x^3 term in the expansion of $(a + 2x)^4$ is 160.

Solution

The x^3 term in the expansion of $(a + 2x)^4$ is

$\binom{4}{3} a(2x)^3 = 160$

$32a = 160$

$\therefore a = 5$

Approximate values

We use expansions to find values of various expressions

Example 13

Expand $(2 + x)^4$ and use your expansion to find

a)$(2.1)^4$

b) $(1.9)^4$

Solution

$$(2 + x)^4 = 2^4 \left(1 + \frac{x}{2}\right)^4$$

$$= 2^4 \left(1 + \binom{4}{1}\left(\frac{x}{2}\right) + \binom{4}{2}\left(\frac{x}{2}\right)^2 + \binom{4}{3}\left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4\right)$$

$$= 16\left(1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16}\right)$$

$$= 16 + 32x + 24x^2 + 8x^3 + x^4$$

(a) $(2.1)^4 = (2 + 0.1)^4; x = 0.1$

$\Longrightarrow (2.1)^4 = 16 + 32(0.1) + 24(0.1)^2 + 8(0.1)^3 + (0.1)^4$

$= 19.4481$

(b) $(1.9)^4 = (2 - 0.1)^4; x = -0.1$

$\Longrightarrow (2.1)^4 = 16 + 32(-0.1) + 24(-0.1)^2 + 8(-0.1)^3 + (-0.1)^4$

$= 13.0321$

Example 14

Expand $(1 + 4x)^{14}$ in ascending powers of x, up to and including the 4th term. Hence evaluate $(1.0004)^{14}$, correct to four decimal places.

Solution

Using the binomial expansion gives

$$(1 + 4x)^{14} = 1 + \binom{14}{1}(4x) + \binom{14}{2}(4x)^2 + \binom{14}{3}(4x)^3 + \cdots$$

$$= 1 + 56x + 1456x^2 + 23296x^3 + \cdots$$

Since $(1.0004)^{14} = [1 + 4(0.0001)]^{14}$

We can use the above expansion with $x = 0.0001$, which gives

$(1.0004)^{14} = 1 + 56(0.0001) + 1456(0.0001)^2 + 23296(0.0001)^3$
$= 1.005614583$
$= 1.0056$ to four decimal places.

Exercise 12a

1) Simplify the following;
(a) $\binom{5}{3}$ (b) $\binom{12}{2}$ (c)` $\binom{7}{3}$ $Ans((a)10\ (b)66\ (c)35)$
2) Expand the following using Pascal's triangle.
a)$(5 + 2x)^3$
b)$(x - \frac{2}{x})^5$
3) Use the binomial theorem to expand $(1 + x)^{12}$in ascending powers of x up to and including in x^3.
4) Find the coefficient of the term indicated in square brackets in the expansion of each of the expressions below.
(a) $(1 + x)^7$ $[x^4]$ (b) $(1 + x)^9$ $[x^2]$ $Ans((a)35\ (b)36)$
5) If x is such that terms involving x^4 and higher powers can be neglected. Find an approximate expansion of $(1 + \frac{x}{2})^{20}$.
6) Expand the following in ascending powers of x.
a) (1+2x)$(1 - x)^{10}$
b) (1+2x-2x^2)$(1 + 2x)^7$
c) $(1 + x + x^2)^6$
d) $(1 + 2x - x^2)^5$.
7) When $(1 + ax)^{10}$ is expanded in ascending powers of x the series expansion is $A + Bx + Cx^2 + 15x^3 + \cdots$.. find the values of A,B,and C $Ans: \left\{\frac{1}{2}, 1{,}5{,}11\frac{1}{4}\right\}$
8) If x is sufficiently small to allow any term in x^5 or higher powers of x to be neglected show that $(1 + x)^6(1 - 2x^3)^{10}$ in 1+6x+15x^2-105x^4.
9) Obtain the expansion of $(1 + x - 2x^2)^8$as far as the term in x^3
10) Find the first three terms of the expansion $(2 - x)^6$and use it to find $(1.998)^6$correct to two decimal places
11) Expand $(1 - 3x + 2x^2)^5$ in ascending powers of x as far as the term in x^2
12) Expand $(1 + 3x)^{10}$ in ascending powers of x up to and including the term in x^3.Hence evaluate $(1.003)^{10}$ correct to 3 places of decimal Ans {1.03041}
13) (a)Expand $(1 + 2x)^{14}$ in ascending powers of x up to and including the term in x^3
(b) Hence evaluate $(1.02)^{14}$ correct to three decimal places.
$Ans\big((a)1 + 28x + 364x^2 + 2912x^3 (b)1.319\big)$
14) By first factorizing the quadratic $3 - 7x - 6x^2$, deduce the first three terms in the binomial expansion of $(3 - 7x - 6x^2)^5$. $Ans[243 - 2835x + 10800x^2]$
15) Use the binomial expansion of $(1 + x)^8$ to the first four terms to find $(1.01)^8$ correct to five significant figures.
16) Expand $\left(1 - \frac{3}{2}x - x^2\right)^5$in ascending powers of x as far as the term in x^4
.*Ans*: $\left(\mathbf{1 - \frac{15}{2}x + \frac{35}{2}x^2 - \frac{15}{4}x^3 - \frac{515}{16}x^4}\right)$

<u>Particular terms of a binomial expansion</u>

- By considering the general term of a binomial expansion a term involving a particular power of x may be found.
- The general term $\mathbf{u_{r+1}}$ is given by $\mathbf{nc_r a^{n-r} x^r}$ which is used to find terms involving particular powers.

Example 15

Find the term involving x^{12} in this expansion of $(3+2x)^{15}$.

Solution

Using the general term in the expansion of

$(a+x)^n$as u_{r+1}=$nc_r a^{n-r}x^r$

Substituting for a=3, 2x for x and n=15,

we have $15c_r(3)^{15-r}(2x)^r$.

To find the term involving x^{12} we need r=12 (in the 13^{th} term)

Thus the required term is $15c_{12}3^3(2x)^{12}$

$$=\frac{15!}{12!3!}3^3 2^{11}x^{12}$$

Example 16

Find the term independent of x in the binomial expansion of $(3x-\frac{2}{x^2})^{18}$

Solution

The general term u_{r+1} in the expansion of $(a+x)^n$ is given by n$c_r a^{n-r}x^r$.

Substituting 3x for a, $(\frac{-2}{x^2})$ for x and n=18

We have u_{r+1}=$18c_r(3x)^{18-r}(\frac{-2}{x^2})^r$

$=18c_r 3^{18}(-2)^r x^{18-3r}$

This term is independent of x if 18-3r=0 i.e. r=6

Thus the required term is u_7=$18C_6 3^{12}(-2)^6$

$=18C_6 3^{12}2^6$

Example 17

Determine n if in the expansion of $(2+3x)^n$ in ascending powers of x, the coefficient of x^{12} is four times that of x^{11}

Solution

Using the general term n$c_r a^r b^{n-r}$, we have the term in x^{12} as $nC_{12}(3x)^{12}(2)^{n-12}$

And the term in x^{11} is $nC_{11}(3x)^{11}(2)^{n-11}$

Equating the coefficients; $nC_{12}(3x)^{12}(2)^{n-12}$=4.$nC_{11}(3x)^{11}(2)^{n-11}$

$\Rightarrow \frac{n!}{(n-12)!12!}.3^{12}.2^{n-11}.2^{-1}=4\frac{n!}{(n-11)!11!}.3^{11}.2^{n-11}$

$\Rightarrow \frac{3}{(n-12)!12\times11!\times2}=\frac{4}{(n-11)(n-12)!11!}$

$\Rightarrow \frac{1}{8}=\frac{4}{(n-11)}$

$\Rightarrow n-11=32$

$\therefore n=43$

Exercise 12b

1. Find the term involving x^8 in the expansion of$(4+x)^{12}$.
2. Find the term independent of x in the expansion of each of the following

a) $\left(3\mathrm{x}+\frac{1}{\mathrm{x}}\right)^{10}$

b) $\left(2x^2+\frac{4}{x}\right)^{12}$

c) $\left(\frac{3}{x^2}-2x\right)^6$

3. In the binomial expansion of $\left(\frac{3}{x^2}-2x\right)^6$ find the coefficient of the;

(a) the term independent of x

(b) x^3

The binomial theorem for any rational index

- $For\,(\mathbf{1}+\boldsymbol{x})^{\boldsymbol{n}}=\mathbf{1}+\binom{\boldsymbol{n}}{\mathbf{1}}\boldsymbol{x}+\binom{\boldsymbol{n}}{\mathbf{2}}\boldsymbol{x}^{\mathbf{2}}+\binom{\boldsymbol{n}}{\mathbf{3}}\boldsymbol{x}^{\mathbf{3}}+\cdots$

  $=\mathbf{1}+\frac{\boldsymbol{nx}}{\mathbf{1}!}+\frac{\boldsymbol{n(n-1)}}{\mathbf{2}!}\boldsymbol{x}^{\mathbf{2}}+\frac{\boldsymbol{n(n-1)(n-2)}}{\mathbf{3}!}\boldsymbol{x}^{\mathbf{3}}+\cdots$

  For all n$\epsilon\mathbb{R}$ provided **-1<x<1** ***i.e.*** $|\boldsymbol{x}|<1$

- **Note** that the condition $|\mathbf{x}|<1$ is necessary because with n factorial or negative the series is infinity and the condition $|\mathbf{x}|<1$ ensures that this infinite series converges.

Example 18

Expand $(1+3\mathrm{x})^{-5}$ in ascending powers of x stating the first four terms and the range of x for which the expansion is valid

Solution

Using $(1+x)^n=1+\frac{nx}{1!}+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$

We substitute n= -5 and x= 3x

$(1+3\mathrm{x})^{-5}=1+\frac{(-5)}{1!}(3x)+\frac{(-5)(-6)}{2!}(3x)^2+\frac{(-5)(-6)(-7)}{3!}(3x)^3+\cdots$

$=1-15x+135x^2-945x^3+\cdots$

The expansion is valid for $|3x|<1\ ie\ |x|<\frac{1}{3}$

Example 19

Expand $\frac{(1+2x)^2}{(2-x)}$ in ascending powers of x up to and including the term in x^3 and state the values for which the expansion is valid

Solution

$\frac{(1+2x)^2}{(2-x)}=(1+2x)^2(2-x)^{-1}$ however, we cannot expand $(2-x)^{-1}$in the form as the binomial expansion for negative and fractional n only applies to $(1+x)^n$, not $(a+x)^n$

However $(2-x)^{-1}=\left[2\left(1-\frac{x}{2}\right)\right]^{-1}$

$=2^{-1}\left(1-\frac{x}{2}\right)^{-1}=\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}$

$\frac{(1+2x)^2}{(2-x)}=(1+2x)^2\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}$

$=(1+4x+4x^2)\frac{1}{2}\left[1+(-1)\left(\frac{-x}{2}\right)+\frac{(-1)(-2)\left(\frac{-x}{2}\right)^2}{2!}+\frac{(-1)(-2)(-3)\left(\frac{-x}{2}\right)^3}{3!}\right]$

$=\frac{1}{2}+\frac{9x}{4}+\frac{25x^2}{8}+\frac{25x^3}{16}+\cdots$

The expansion is valid for $\left|\frac{-x}{2}\right| < 1\ i.e\ |x| < 2$

Example 20

Expand $(3+x)^{-2}$ in ascending powers of $\frac{1}{x}$ stating the first four terms only and the values of x for which the expansion is valid

Solution

$$(3+x)^{-2} = \left[x\left(\frac{3}{x}+1\right)\right]^{-2}$$
$$=\frac{1}{x^2}\left(1+\frac{3}{x}\right)^{-2}$$
$$=\frac{1}{x^2}\left\{1+(-2)\left(\frac{3}{x}\right)+\frac{(-2)(-3)}{2!}\left(\frac{3}{x}\right)^2+\cdots\right\}$$
$$=\frac{1}{x^2}-\frac{6}{x^3}+\frac{27}{x^4}-\frac{108}{x^5}+\cdots$$

The expansion is valid for $\left|\frac{3}{x}\right| < 1$ i.e. $|x| > 3$

Example 20

Find the first four terms in the expansion of $(1-8x)^{\frac{1}{2}}$in the ascending powers of x. Substitute $x = \frac{1}{100}$and obtain the value of $\sqrt{23}$correct to five decimal places.

Solution

$$(1-8x)^{\frac{1}{2}} = 1+\frac{\left(\frac{1}{2}\right)}{1!}(-8x)+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}(-8x)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{3!}(-8x)^3+\cdots$$
$$= 1-4x-8x^2-32x^3+\cdots$$
$$\Rightarrow \left(1-\frac{8}{100}\right)^{\frac{1}{2}} \approx 1-4\left(\frac{1}{100}\right)-8\left(\frac{1}{100}\right)^2-32\left(\frac{1}{100}\right)^3$$
$$\Rightarrow \sqrt{\left(\frac{92}{100}\right)} \approx 0.959168$$
$$\Rightarrow \sqrt{\left(\frac{4\times 23}{100}\right)} \approx 0.959168$$
$$\Rightarrow \frac{2}{10}\sqrt{23} \approx 0.959168$$

$\therefore \sqrt{23} \approx 4.79584$ to five decimal places.

Example 21

Obtain the first four terms in the expansion of $\frac{1}{1+x}$

Solution

Now

$$\frac{1}{1+x} = (1+x)^{-1}$$

Using the binomial expansion gives

$$(1+x)^{-1} = 1+(-1)\text{x}+\frac{(-1)(-1-1)}{2!}\text{x}^2+\frac{(-1)(-1-1)(-1-2)}{3!}\text{x}^3+\cdots$$
$$= 1-x+x^2-x^3+\cdots \text{ valid for } |x| < 1$$

Example 22

Obtain the expansion of $\frac{(1+x)^3}{(2-x)}$ up to and including the term in x^3. Hence evaluate $(1.2)^3$ correct to 2 decimal places.

Solution

Now

$\frac{(1+x)^3}{(2-x)} = (1+x)^3(2-x)^{-1}$

Expanding $(1+x)^3$ using the binomial theorem gives

$(1+x)^3 = 1 + \binom{3}{1}x + \binom{3}{2}x^2 + x^3$

$= 1 + 3x + 3x^2 + x^3$

Expanding $(2-x)^{-1}$ using the binomial theorem gives

$(2-x)^{-1} = 2^{-1}\left(1 - \frac{x}{2}\right)^{-1}$

$= \frac{1}{2}\left(1 + \left(\frac{-x}{2}\right)\right)^{-1}$

$= \frac{1}{2}\left(1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-1-1)}{2!}\left(-\frac{x}{2}\right)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}\left(-\frac{x}{2}\right)^3\right)$

$= \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots\right)$

$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots$ valid for $\left|\frac{x}{2}\right| < 1$ $i.e. |x| < 2$

Therefore, we have

$(1+x)^3(2-x)^{-1} = (1 + 3x + 3x^2 + x^3)\left(\frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \cdots\right)$

$= \frac{1}{2} + \frac{7x}{4} + \frac{19x^2}{8} + \frac{27x^3}{16} + \cdots$

$\therefore \frac{(1+x)^3}{(2-x)} = \frac{1}{2} + \frac{7x}{4} + \frac{19x^2}{8} + \frac{27x^3}{16} + \cdots$ valid for $|x| < 2$

Let $x = 0.2$ (which lies in the valid range) then,

$\frac{(1+0.2)^3}{(2-0.2)} \approx \frac{1}{2} + \frac{7(0.2)}{4} + \frac{19(0.2)^2}{8} + \frac{27(0.2)^3}{16}$

$\frac{(1.2)^3}{(1.8)} \approx 0.9585$

$\therefore (1.2)^3 \approx 1.8 \times 0.9585$

≈ 1.73 (2 decimal places)

Example 23

Show that if x is small enough for its cube and higher powers to be neglected, $\sqrt{\frac{1-x}{1+x}} = 1 - x + \frac{x^2}{2}$ by putting $x = \frac{1}{8}$ show that $\sqrt{7} \approx 2\frac{83}{128}$

Solution

Now

$\sqrt{\frac{1-x}{1+x}} = (1-x)^{\frac{1}{2}}(1+x)^{\frac{-1}{2}}$

Expanding $(1-x)^{\frac{1}{2}}$ using the binomial theorem gives

$(1-x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)(-x) + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-x)^2 + \cdots$

$= 1 - \frac{x}{2} - \frac{1}{8}x^2 + \cdots$

Expanding $(1+x)^{\frac{-1}{2}}$ using the binomial theorem gives

$(1+x)^{\frac{-1}{2}} = 1 + \left(\frac{-1}{2}\right)(x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(x)^2 + \cdots$

$= 1 - \frac{x}{2} + \frac{3}{8}x^2 + \cdots$

Therefore, we have

$(1-x)^{\frac{1}{2}}(1+x)^{\frac{-1}{2}} = \left(1 - \frac{x}{2} - \frac{3}{8}x^2 + \cdots\right)\left(1 - \frac{x}{2} + \frac{3}{8}x^2 + \cdots\right)$

$= 1 - x + \frac{x^2}{2}$

$\therefore \sqrt{\frac{1-x}{1+x}} = 1 - x + \frac{x^2}{2}$

Using $x = \frac{1}{8}$

$\Rightarrow \sqrt{\frac{1-\left(\frac{1}{8}\right)}{1+\left(\frac{1}{8}\right)}} \approx 1 - \left(\frac{1}{8}\right) + \frac{\left(\frac{1}{8}\right)^2}{2}$

$\Rightarrow \sqrt{\frac{7}{9}} \approx 1 - \left(\frac{1}{8}\right) + \left(\frac{1}{128}\right)$

$\therefore \sqrt{7} \approx 3 \times \left(\frac{113}{128}\right)$

$\approx 2\frac{83}{128}$

Exercise 12c

1) Show that if x is small enough for its cube and higher powers to be neglected, $\sqrt{\frac{1+x}{1-x}} = 1 + x + \frac{x^2}{2}$by putting $x = \frac{1}{9}$show that $\sqrt{5} \approx \frac{181}{81}$
2) Find the first four terms in the expansion of $(1+3x)^{\frac{1}{3}}$ in ascending powers o f x and state the range of values of x for which the expansion is valid.
3) In the expansion of $(1+ax)^n$, the first three terms are $1 - \frac{5}{2}x + \frac{75}{8}x^2$, find n and a and state the range of values of x for which the expansion is valid.
4) Find the coefficient of x^{10} in the expansion of $(2x-3)^{14}$
5) Expand $(1+x)^{\frac{1}{2}}$ in ascending powers of x as far as the term in x^2 and hence find an approximation for$\sqrt{1.08}$. Deduce that $\sqrt{12} = 3.464$.
6) (a)Find the values of the constants a and b for which the expansion in ascending powers of x of the two expansions $(1+2x)^{\frac{1}{2}}$ and $\frac{1+ax}{1+bx}$ up to and including the term in x^2 are the same

(c) With these values of a and b use the result $(1+2x)^{\frac{1}{2}} \approx \frac{1+ax}{1+bx}$ with $x = \frac{-1}{100}$ to obtain an approximate value for $\sqrt{2}$ in the form $\frac{p}{q}$ where p and q are positive integers.

7) For each of the following expand up to the term in x^3

$\sqrt{\left(\frac{1-x}{1+2x}\right)}$ b) $\sqrt{\left(\frac{1+x}{1-x}\right)}$

8) Use the binomial theorem to find $\sqrt{1.05}$ to four places of decimals.

9) Expand $\frac{\sqrt{(1+3x-4x^2)}}{(1-2x)^2}$ in ascending powers of x as far as the term in x^3, assuming that the value of x is such that the expansion converges.

10) In the expansion of $(1+ax+2x^2)^6$ in powers of x, the coefficients of x^2 and x^{11} are 27 and -192 respectively. Find a and the coefficient of x^3 and x^{10}.

11) Find the value of n for which the coefficients of x, x^2 and x^3 in the expansion of $(1+x)^n$ are in arithmetical progression. $Ans(n=8)$

12) Find the coefficient of x^{17} in the expansion of $\left(x^3+\frac{1}{x^4}\right)^{15}$ (Ans:1365)

13) Show that, if x is so small that x^4 and higher powers of x can be neglected. Then $\left\{\left(1+\frac{x}{2}\right)^3-(1+3x)^{\frac{1}{2}}\right\} \div \left(1-\frac{5}{6}x\right) = \mathbf{\frac{15}{8}x^2}$

14) Write down and simplify the term independent of x in the expansion of $\left(3x^2-\frac{1}{2x}\right)^9$.

15) If x is so small that x^3 and higher powers can be neglected, show that $\left(\mathbf{1}-\frac{\mathbf{3}}{\mathbf{2}}\boldsymbol{x}\right)^{\mathbf{5}}(\mathbf{2}+\mathbf{3}\boldsymbol{x})^{\mathbf{6}} = \mathbf{64}+\mathbf{96}\boldsymbol{x}-\mathbf{720}\boldsymbol{x}^{\mathbf{2}}$

16) Find by binomial theorem the coefficient of x^8 in the expansion of $(3-5x^2)^{\frac{1}{2}}$ in ascending powers of x.

17) Given that x is so small that x^3 and higher powers of x can be neglected, show that $\frac{1}{\sqrt{1+x}} = 1-\frac{x}{2}+\frac{3x^2}{8}$. By letting $x=\frac{1}{4}$, find a rational approximation of $\sqrt{5}$. $Ans\left[\frac{256}{115}\right]$

18) (a) Show that $\left(1+\frac{x}{25}\right)^{\frac{1}{2}} \equiv 1+\frac{x}{50}-\frac{x^2}{5000}+\frac{x^3}{250000}+\cdots$

(d) By substituting $x=1$ into your answer to part (a), deduce that $\sqrt{26} \approx 5.09902$

Use of partial fractions in Binomial expansion

Partial fractions are used to simplify expressions before using the binomial theorem

Example 24

Express $f(x)=\frac{4-3x}{(1-x)(2-x)}$ and hence expand $f(x)$ using binomial theorem in the ascending powers of x up to the term in x^2 and state the range of values of x for which the expansion is valid.

Solution

$f(x)=\frac{4-3x}{(1-x)(2-x)}=\frac{1}{1-x}+\frac{2}{2-x}$

Re-arranging $\frac{1}{1-x}+\frac{2}{2-x}$ as $\frac{1}{1-x}+\frac{1}{1-\frac{1}{2}x}$ or $(1-x)^{-1}+\left(1-\frac{1}{2}x\right)^{-1}$ in order to make each term to be expanded begin with unity.

$\Rightarrow (1-x)^{-1}=1+x+x^2+\cdots$

$\Rightarrow \left(1-\frac{1}{2}x\right)^{-1}=1+\frac{3}{2}x+\frac{5}{4}x^2+\cdots$

$\therefore (1-x)^{-1}+\left(1-\frac{1}{2}x\right)^{-1}=2+\frac{3}{2}x+\frac{5}{4}x^2+\cdots$

The binomial theorem is valid since each term begins with unity provided that $\mathbf{x}$ and $\frac{\mathbf{1}}{\mathbf{2}}\mathbf{x}$ lie between $\pm\mathbf{1}$. The expression is therefore valid provided x lies between $\pm\mathbf{1}$.

Example 25

Expand $\frac{7+x}{(1+x)(1+x^2)}$ in ascending powers of x as far as the term in x^4 and state the range for which the expansion is valid.

Solution

$\Rightarrow \frac{7+x}{(1+x)(1+x^2)} = \frac{3}{1+x} + \frac{4-3x}{1+x^2}$

$=3(1+x)^{-1} + (4-3x)(1+x^2)^{-1}$

$=3(1-x+x^2-x^3+x^4-\cdots) + (4-3x)(1-x^2+x^4-x^6+\cdots)$

$=7-6x-x^2+7x^4+\cdots$

$\therefore \frac{7+x}{(1+x)(1+x^2)} = 7-6x-x^2+7x^4+\cdots$

the series being convergent if $-1 < x < 1$

Exercise 12d

1. Expand the following using binomial theorem up to the term in x^3 and state the range of values of x for which the expansion is valid.

   (a) $\frac{(1-x)^2}{(1-2x)^3}$

   (b) $\frac{x}{(1+x)(1-x)^2}$
2. Express in partial fractions $\frac{x^3+6x^2+17x+4}{(x^2+1)(6+x-2x^2)}$. Show that, neglecting x^3 and higher powers of x, the expression is equal to $\frac{1}{3}\left(2+\frac{49}{6}x+\frac{11}{36}x^2\right)$
3. Express $\frac{4x+1}{(x+1)^2(x-1)}$ in its partial fractions. Hence obtain the first four terms in the expansion of the expression, stating the necessary restrictions on the value of x.
4. Express $\frac{2x^3}{(1+x^2)(1-x)^2}$ as a sum of three partial fractions; and obtain an expansion in ascending powers of x, of this expression as far as the term in x^7
5. Express $\frac{5x-1}{(x+1)^2(x-2)}$ in partial fractions. Hence obtain the first four terms in the expansion of this expression in ascending powers of x, stating the necessary restrictions on the value of x

END